Solution of Rolewicz’s Problem

Solution of Problem 01-005 by BOGDAN CHOCZEWSKI (Faculty of Applied Mathematics, University of Mining and Metallurgy (AGH), Krakow, Poland), ROLAND GIRGENSOHN (Institute of Biomathematics and Biometry, GSF-Forschungszentrum, Neuherberg, Germany), and ZYGFRYD KOMINEK (Institute of Mathematics, Silesian University, Katowice, Poland).

1. Problem. We shall solve the following problem.

Problem (P). (Rolewicz). Find all nonnegative and differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying the inequality

\[
(P) \quad f(t) - f(s) - f'(s)(t-s) \geq f(t-s), \quad t, s \in \mathbb{R}
\]

(cf. [2] and [4], where the problem was originally stated, under the additional assumption that \( f \) be even).

It turns out that the assumption is not needed; every solution of Problem (P) is automatically a quadratic function (and therefore even).

We also find all pairs \((f, g)\), \( f, g : \mathbb{R} \to \mathbb{R} \), satisfying the functional inequality obtained from (P) by replacing \( f'(s) \) by \( g(s) \) as well as those which satisfy the related functional equation (without any regularity assumptions on \( f \) and \( g \)).

2. Solution. We are going to prove the following theorem.

Theorem 1. The only solutions \( f : \mathbb{R} \to \mathbb{R} \) of problem (P) are given by the formula

\[
(S) \quad f(x) = Cx^2, \quad x \in \mathbb{R},
\]

where \( C \) is a nonnegative constant.

Proof. If a function \( f : \mathbb{R} \to \mathbb{R} \) is a solution to (P), then

\[
f(0) = f'(0) = 0
\]

(put \( t = s = 0 \) in (P) to get \( f(0) = 0 \) and then \( s = 0 \) in (P) to obtain \( f'(0)t \leq 0, \ t \in \mathbb{R} \), yielding \( f'(0) = 0 \)). Thus

\[
(1) \quad \lim_{s \to 0} \frac{f(s)}{s} = 0.
\]

Denote \( h := t - s \in \mathbb{R} \) and rewrite (P) as

\[
(2) \quad f'(s) \cdot h \leq f(s + h) - f(s) - f(h), \quad s, h \in \mathbb{R}.
\]
Now assume $s > 0$ to get that

$$\frac{f'(s)}{s}h \leq \frac{f(s + h) - f(h)}{s} - \frac{f(s)}{s}, \quad h \in \mathbb{R}, s > 0. \quad (3)$$

Thanks to (1), when $s \to 0^+$, the RHS tends to $f'(h)$. Thus, the LHS is bounded from above, and

$$2C := \limsup_{s \to 0^+} \frac{f'(s)}{s}$$

exists. From (3) we get

$$2Ch \leq f'(h), \quad h \in \mathbb{R}. \quad (4)$$

Now assume that $s < 0$ in (2), which gives us

$$\frac{f'(s)}{s}h \geq \frac{f(s + h) - f(h)}{s} - \frac{f(s)}{s}, \quad h \in \mathbb{R}, s < 0. \quad (5)$$

As before, this implies that

$$2D := \liminf_{s \to 0^-} \frac{f'(s)}{s}$$

exists and that

$$2Dh \geq f'(h), \quad h \in \mathbb{R}. \quad (5)$$

Inequalities (4) and (5) together now imply that $Ch \leq Dh$ for all $h \in \mathbb{R}$, and thus $C = D$. Now using (4) and (5) once more, we have $f'(h) = 2Ch$ for all $h \in \mathbb{R}$, and taking $f(0) = 0$ into account we get (S), which was to be proved. \[\square\]

**Remark.** The proof of Theorem 1 is due to the second author. Earlier the other authors had proved (S) with the aid of the following proposition.

**Proposition 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be an even, nonnegative, and differentiable function with $f(1) = 1$, satisfying inequality (P). Then we have the following assertions.

(a) Either $f$ is given by (S) with $C = 1$ or there are an $\varepsilon > 0$ and $a, b \in \mathbb{R}$, $\frac{1}{2} < a < b$, such that $f'(x) > 2x + \varepsilon$, $x \in [a, b]$.

(b) If there exists a sequence $(x_n)_{n \in \mathbb{N}}$ converging to zero, $x_n > 0$, $n \in \mathbb{N}$, such that $f'(x_n) \geq 2x_n$, $n \in \mathbb{N}$, then $f$ is given by (S) with $C = 1$. 
The first author was able to derive (S) from (P) having additionally assumed that $f$ is even, twice differentiable in a neighborhood of the origin, and it satisfies an initial condition; cf. [1].

3. Pexider-type functional inequality. In connection with (P) let us consider the following inequality:

\[(Q) \quad f(t) - f(s) - g(s)(t - s) \geq f(t - s), \quad t, s \in \mathbb{R}.\]

We start with a simple lemma.

**Lemma 1.** A pair $(f, g)$ of functions, each mapping $\mathbb{R}$ into $\mathbb{R}$, where $f$ is differentiable in $\mathbb{R}$, $f(0) = 0$, and $g$ is arbitrary, satisfies inequality (Q) if and only if

\[(6) \quad g(t) = f'(t) - f'(0), \quad t \in \mathbb{R},\]

and $f$ satisfies the inequality

\[(P') \quad f(t) - f(s) - [f'(s) - f'(0)](t - s) \geq f(t - s), \quad t, s \in \mathbb{R}.\]

**Proof.** Let $f$ and $g$, regular as required, satisfy (Q). For $t > s$ inequality (Q) may be written in the form

\[(7) \quad \frac{f(t) - f(s)}{t - s} - g(s) \geq \frac{f(t - s)}{t - s}\]

whereas for $t < s$ we have the inequality opposite to (7). Since $f$ is differentiable, we get $f'(s) - g(s) = f'(0)$, which is (6), and $f$ satisfies (P'). The converse implication is obvious. \(\square\)

Theorem 1 and Lemma 1 together yield the following result.

**Theorem 2.** If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a nonnegative and differentiable function with $f'(0) = 0$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is arbitrary, and they both satisfy inequality (Q), then there is a $C \geq 0$ such that

\[f(t) = C t^2, \quad g(t) = 2C t, \quad t \in \mathbb{R}.\]

In the case where $f$ in (Q) is an odd function we have the following theorem.

**Theorem 3.** A pair $(f, g)$ of functions, each mapping $\mathbb{R}$ into $\mathbb{R}$, where $f$ is differentiable in $\mathbb{R}$ and odd, and $g$ is arbitrary, satisfies inequality (Q) if and only if there is a $C \in \mathbb{R}$ such that

\[(8) \quad f(t) = C t, \quad g(t) = 0, \quad t \in \mathbb{R}.\]
Proof. We have \( f(0) = 0 \) as \( f \) is odd. Thus the lemma works. Since now \( f' \) in \((P')\) is even, on putting \(-s\) in place of \(s\) in \((P')\) we get

\[
f(t) + f(s) - [f'(s) - f'(0)](t + s) \geq f(t + s), \quad s, t \in \mathbb{R}.
\]

With \( t = 0 \) here we arrive at \([f'(s) - f'(0)] \cdot s \leq 0, s \in \mathbb{R}\).

On the other hand, with \(-t\) in place of \(t\) in \((P')\) we obtain

\[
[f'(s) - f'(0)](t + s) \geq f(t) + f(s) - f(t + s), \quad t, s \in \mathbb{R}.
\]

Letting \( t = 0 \) here yields \([f'(s) - f'(0)] \cdot s \geq 0, s \in \mathbb{R}\).

Consequently, \( f'(s) = f'(0) \), in turn \( f(s) = f'(0)s + B \). But \( B = 0 \) as \( f \) is odd. Finally, by (6), \( g(s) = 0, s \in \mathbb{R} \). Thus (8) holds with \( C = f'(0) \). The converse implication is obvious. \( \Box \)

4. Pexider-type functional equation. For the functional equation (cf. inequality (Q))

\[
(E) \quad f(t) - f(s) - g(s)(t - s) = f(t - s), \quad t, s \in \mathbb{R},
\]

we have the following result.

**Theorem 4.** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be functions fulfilling equation \((E)\). Then there exist a real constant \( C \) and an additive function \( a : \mathbb{R} \to \mathbb{R} \) such that

\[
(9) \quad f(x) = a(x) + Cx^2, \quad g(x) = 2Cx, \quad x \in \mathbb{R}.
\]

Conversely, the system of functions defined by \((9)\), where \( a \) is an additive function and \( C \in \mathbb{R} \), is a solution of \((E)\).

**Proof.** Setting \( s = 0 \) in \((E)\) we get

\[
f(0) = g(0) = 0.
\]

Put \( t + s \) instead of \( t \) in \((E)\). We have

\[
(10) \quad f(t + s) - f(t) - f(s) = g(s)t, \quad t, s \in \mathbb{R}.
\]

Since the LHS of this equality is symmetric with respect to \( t \) and \( s \), so is its RHS. Thus

\[
g(s)t = g(t)s, \quad t, s \in \mathbb{R}.
\]

Therefore there exists a constant \( C \in \mathbb{R} \) such that \( g(x) = 2Cx, \ x \in \mathbb{R} \). Moreover, now \((10)\) has the form

\[
(11) \quad f(t + s) - f(t) - f(s) = 2Cts, \quad t, s \in \mathbb{R}.
\]
We define the function \( a : \mathbb{R} \rightarrow \mathbb{R} \) by the formula
\[
a(x) := f(x) - Cx^2, \quad x \in \mathbb{R}.
\]

According to (11) we obtain
\[
a(t + s) - a(t) - a(s) = 2Cts - C(t + s)^2 + Cs^2 + Ct^2 = 0
\]
for all \( t, s \in \mathbb{R} \), which means that \( a \) is an additive function. The other part of the proof is evident.

Since every Lebesgue measurable additive function \( a : \mathbb{R} \rightarrow \mathbb{R} \) is linear (cf. [3], for example), Theorem 4 has the following corollary.

**Corollary 1.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a Lebesgue measurable function, and let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be an arbitrary function. Then the pair of function \((f, g)\) is a solution of functional equation \((E)\) if and only if there exist a real constants \( C \) and \( b \) such that
\[
f(x) = Cx^2 + bx, \quad g(x) = 2Cx, \quad x \in \mathbb{R}.
\]

**REFERENCES**


