A Calculus Exam Misprint Ten Years Later

In *Problem* 99-005 Michael Renardy (Virginia Tech) *and* Thomas Hagen (University of Memphis) raised the following question: does the series

$$\sum_{n=1}^{\infty} \frac{(2+\sin n)^n}{3^n\, n}$$

converge? This question had been given on a calculus exam by mistake.

Solution by Albert Stadler[1] (Herrliberg, Switzerland).

A heuristic argument runs as follows. Note that

$$\frac{(2+\sin n)^n}{n\, 3^n} = \frac{1}{n}\left(1 - \frac{1-\sin n}{3}\right)^n.$$

So the main contributions to the sum

$$\sum_{n=1}^{\infty} \frac{(2+\sin n)^n}{n\, 3^n}$$

stem from those integers n for which $\sin n$ is close to 1, which occurs if and only if n is close to $(2q+\frac{1}{2})\pi$, where q is a nonnegative integer. However, the sequence $\{(2q+\frac{1}{2})\pi\}_{q\geq 0}$ is equidistributed mod 1, because π is irrational. So $\sin n$ is close to 1 only for a small portion of n's, too small actually to make the sum diverge. Now let's turn this heuristic argument into a proof!

Let $0<\varepsilon<\frac{1}{4}$ and let $\psi(x)$ be an even function that is 10 times continuously differentiable on $(-\infty,\infty)$ and satisfies $\psi(x)=1$ for $|x|\leq 1$, $\psi(x)=0$ for $|x|\geq 2$, and $\psi(x)\geq 0$ for all real x. An instance of such a function is

$$\psi(x) = \begin{cases} 1, & |x|\leq 1, \\ \dfrac{21!}{(10!)^2}\displaystyle\int_{|x|}^{2}(t-1)^{10}(2-t)^{10}\,dt, & 1<|x|<2, \\ 0, & |x|\geq 2. \end{cases}$$

It is readily verified that

(i) $\psi(x)=\psi(-x)$ for all x,

[1]E-mail: albert.stadler@ubs.com

(ii) $\psi(x) \geq 0$ for all x,

(iii) $\psi^{(k)} = 0,\ |x| < 1$ or $|x| > 2,\ k \geq 1$,

(iv) $\lim_{x\to 1} \psi(x) = \frac{21!}{(10!)^2} \int_1^2 (t-1)^{10}(2-t)^{10}\, dt = 1$,

$\lim_{x\to 2} \psi(x) = 0$,

(v) $\psi^{(k)}(t) = -\frac{21!}{(10!)^2} \frac{d^{k-1}}{dt^{k-1}} \left[(t-1)^{10}(2-t)^{10}\right]$

$= -\frac{21!}{(10!)^2} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{10!}{(10-j)!} \frac{10!}{(10-k+1+j)!} (t-1)^{10-j}(t-2)^{10-k+1+j}, \quad 1 < t < 2,$

so $\lim_{t\to 1} \psi^{(k)}(t) = \lim_{t\to 2} \psi^{(k)}(t) = 0, \quad 1 \leq k \leq 10$.

Put $\phi(x) = \psi(x/\varepsilon)$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and extend $\phi(x)$ to a 1-periodic function on the whole real line. Then $\phi(x)$ can be expanded into a Fourier series

$$\phi(x) = \sum_{n=-\infty}^{\infty} a_n\, e^{2\pi i n x},$$

whose coefficients a_n are given by

$$a_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x)\, e^{-2\pi i n x}\, dx.$$

The condition on differentiability implies that if $n \neq 0$,

$$a_n = \frac{1}{(2\pi i n)^{10}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi^{(10)}(x)\, e^{-2\pi i n x}\, dx,$$

which follows from the original representation of a_n by 10-fold partial integration. Clearly, $\phi^{(10)}(x) = \varepsilon^{-10}\psi^{(10)}(x/\varepsilon)$. So, if $n \neq 0$,

$$(1) \qquad |a_n| \leq \frac{1}{(2\pi n)^{10}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\phi^{(10)}(x)\right| dx = \frac{1}{(2\pi n\varepsilon)^{10}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\psi^{(10)}(x/\varepsilon)\right| dx$$

$$= \frac{1}{(2\pi n)^{10}\varepsilon^9} \int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}} \left|\psi^{(10)}(x)\right| dx = \frac{1}{(2\pi n)^{10}\varepsilon^9} \int_{-2}^{2} \left|\psi^{(10)}(x)\right| dx = \frac{c_1}{\varepsilon^9 n^{10}}$$

for some absolute constant c_1. For $n = 0$ we have

$$(2) \qquad |a_0| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |\phi(x)| dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\psi(x/\varepsilon)| dx = \varepsilon \int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}} |\psi(x)| dx = \varepsilon \int_{-2}^{2} |\psi(x)| dx = c_2\, \varepsilon.$$

Put

$$\delta = \frac{1 - \cos \varepsilon}{3} = \frac{2 \sin^2(\varepsilon/2)}{3}.$$

Then $\sin n \in (1 - 3\delta, 1]$ if and only if there is an integer q such that $\phi((4q + 1)\pi/2) = 1$. This is because $\sin n = \cos(n - \pi/2) \in (1 - 3\delta, 1]$ implies that there is an integer q such that $|n - (4q + 1)\pi/2| < \varepsilon$ and therefore $\phi((4q + 1)\pi/2) = 1$, and vice versa. Given a positive integer N, put

$$\sigma_N := \sum_{n=N+1}^{2N} \frac{(2 + \sin n)^n}{n\, 3^n}.$$

Let

$$A = \{n | N < n \leq 2N \text{ and } 1 - \sin n < 3\delta\},$$

$$A' = \{n | N < n \leq 2N \text{ and } 1 - \sin \geq 3\delta\}.$$

Then

(3)
$$\begin{aligned} 0 < \sigma_N &< \frac{1}{N} \sum_{n=N+1}^{2N} \left(1 - \frac{1 - \sin n}{3}\right)^n \\ &= \frac{1}{N} \sum_{n \in A} \left(1 - \frac{1 - \sin n}{3}\right)^n + \frac{1}{N} \sum_{n \in A'} \left(1 - \frac{1 - \sin n}{3}\right)^n \\ &< \frac{1}{N} \sum_{n \in A} 1 + (1 - \delta)^N \leq \frac{1}{N} \sum_{n \in A} 1 + \exp(-\delta N). \end{aligned}$$

Let $\|x\| = \min_{n \in Z} |x - n|$ denote the distance from x to the nearest integer. Let a, b be integers, $a < b$. If x is not an integer, then

$$\left| \sum_{n=a}^{b} e^{2\pi i n x} \right| = \left| e^{2\pi i a x} \sum_{n=0}^{b-a} e^{2\pi i n x} \right| = \left| \frac{e^{2\pi i (b-a+1)x} - 1}{e^{2\pi i x} - 1} \right| \leq \frac{2}{|e^{i\pi x} - e^{-i\pi x}|} = \frac{1}{|\sin \pi x|} \leq \frac{1}{2\|x\|}.$$

Let

$$B = \left\{ n \,\middle|\, N < n \leq 2N \text{ and } \left| n - \frac{\pi}{2}(4q + 1) \right| < \varepsilon \text{ for some } q \in \mathbb{Z} \right\},$$

$$\begin{aligned} C &= \left\{ q \,\middle|\, N + 1 - \frac{1}{4} \leq \frac{\pi}{2}(4q + 1) \leq 2N + \frac{1}{4} \right\} \\ &= \left\{ q \,\middle|\, N + 1 - \frac{1}{4} - \frac{\pi}{2} \leq 2\pi q \leq 2N + \frac{1}{4} - \frac{\pi}{2} \right\}. \end{aligned}$$

Then

$$\frac{1}{N}\sum_{n\in A}1=\frac{1}{N}\sum_{n\in B}1\le\frac{1}{N}\sum_{q\in C}\phi\left(\frac{\pi}{2}(4q+1)\right) \tag{4}$$

$$=\frac{1}{N}\sum_{k=-\infty}^{\infty}a_k\,e^{i\pi^2k}\sum_{q\in C}e^{4i\pi^2qk}$$

$$\le\frac{1}{N}|a_0|\left(\frac{N}{2\pi}+1\right)+\frac{1}{2N}\sum_{k\ne0}\frac{|a_k|}{\|2\pi k\|}$$

$$\le c_3\,\varepsilon+\frac{c_1}{2N}\sum_{k\ne0}\frac{1}{\varepsilon^9k^{10}\|2\pi k\|},$$

where we have used the estimates (1) and (2).

Now we refer to a theorem of Hata [1] on rational approximations to π that implies $\|2n\pi\|\ge n^{-8}$ for all sufficiently large n. *Note.* In fact, Hata proves that

$$|q\pi-p|\ge\frac{1}{q^{7.0161}}$$

for integers p,q with $q>q_0$, so $\|2n\pi\|\ge n^{-8}$ holds with ample room to spare. This allows us to bound the sum in (4) and obtain

$$\frac{1}{N}\sum_{A}1\le c_3\,\varepsilon+\frac{c_1}{2N}\sum_{k\ne0}\frac{1}{\varepsilon^9k^2}\le c_3\,\varepsilon+\frac{c_4}{N\varepsilon^9}. \tag{5}$$

To sum up, by (3) and (5),

$$0<\sigma_N<c_3\,\varepsilon+\frac{c_4}{N\varepsilon^9}+\exp(-\delta N)\le c_3\,\varepsilon+\frac{c_4}{N\varepsilon^9}+\exp(-c_5\,\varepsilon^2N).$$

Now we choose $\varepsilon=N^{-1/10}$ and obtain $0<\sigma_N<c_6N^{-1/10}$. This concludes the proof because

$$\sum_{n=2^K+1}^{\infty}\frac{(2+\sin n)^n}{n\,3^n}=\sum_{j=K}^{\infty}\sigma_{2^j}\le c_6\sum_{j=K}^{\infty}2^{-j/10}<\infty.$$

We thank the anonymous referee for some useful comments that made the text more readable.

REFERENCE

[1] M. Hata, *Rational approximations to π and some other numbers*, Acta Arith., 63 (1993), pp. 335–349.