The Kronecker Product

A Product of the Times

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The Kronecker Product

$B \otimes C$ is a *block matrix* whose ij-th block is $b_{ij}C$.

E.g.,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \otimes C \;=\; \left[\begin{array}{c|c} b_{11}C & b_{12}C \\ \hline b_{21}C & b_{22}C \end{array}\right]$$

Also called the "Direct Product" or the "Tensor Product"

Every $b_{ij}c_{kl}$ Shows Up

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \otimes \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$=$$

$$\left[\begin{array}{ccc|ccc} b_{11}c_{11} & b_{11}c_{12} & b_{11}c_{13} & b_{12}c_{11} & b_{12}c_{12} & b_{12}c_{13} \\ b_{11}c_{21} & b_{11}c_{22} & b_{11}c_{23} & b_{12}c_{21} & b_{12}c_{22} & b_{12}c_{23} \\ b_{11}c_{31} & b_{11}c_{32} & b_{11}c_{33} & b_{12}c_{31} & b_{12}c_{32} & b_{12}c_{33} \\ \hline b_{21}c_{11} & b_{21}c_{12} & b_{21}c_{13} & b_{22}c_{11} & b_{22}c_{12} & b_{22}c_{13} \\ b_{21}c_{21} & b_{21}c_{22} & b_{21}c_{23} & b_{22}c_{21} & b_{22}c_{22} & b_{22}c_{23} \\ b_{21}c_{31} & b_{21}c_{32} & b_{21}c_{33} & b_{22}c_{31} & b_{22}c_{32} & b_{22}c_{33} \end{array}\right]$$

Basic Algebraic Properties

$$
\begin{aligned}
(B \otimes C)^T &= B^T \otimes C^T \\
(B \otimes C)^{-1} &= B^{-1} \otimes C^{-1} \\
(B \otimes C)(D \otimes F) &= BD \otimes CF \\
B \otimes (C \otimes D) &= (B \otimes C) \otimes D \\
C \otimes B &= (\text{Perfect Shuffle})^T (B \otimes C)(\text{Perfect Shuffle})
\end{aligned}
$$

Reshaping KP Computations

Suppose $B, C \in {\rm I\!R}^{n \times n}$ and $x \in {\rm I\!R}^{n^2}$.

The operation $y = (B \otimes C)x$ is $O(n^4)$:

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y = kron(B,C)*x
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The operation $Y = CXB^T$ is $O(n^3)$:

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y = reshape(C*reshape(x,n,n)*B',n,n)
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Talk Outline

1. **The 1800's**

   Origins: $\textcircled{Z}$

2. **The 1900's**

   Heightened Profile: $\otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes$

3. **The 2000's**

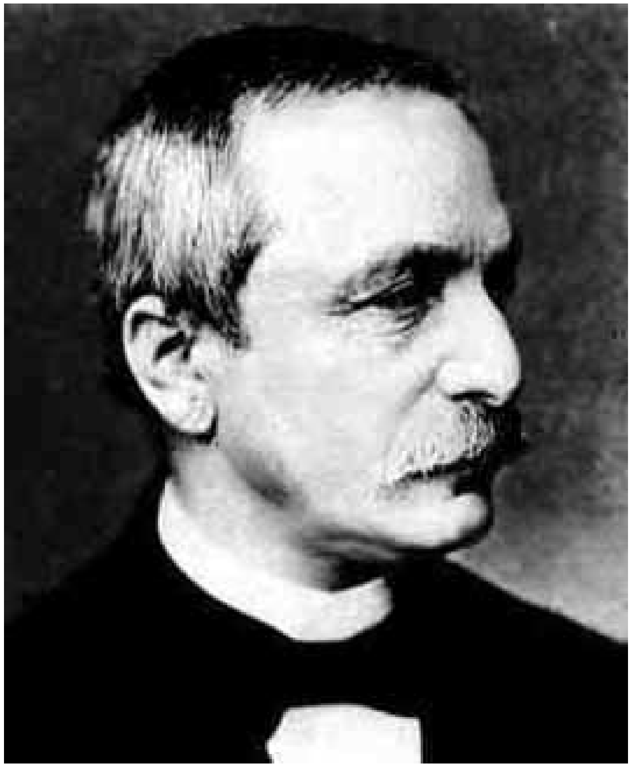
   Future: $\textcircled{\infty}$

The 1800’s

Z

Products and Deltas

$\bigotimes$



δ_{ij}

Leopold Kronecker (1823–1891)

The Kronecker Delta: Not as Interesting!

$$U^T \delta_{ij} V = |\delta_{ij}|$$

$$\kappa_2(\delta_{ij}) = \frac{1}{\delta_{ij}}$$

Acknowledgement

H.V. Henderson, F. Pukelsheim, and S.R. Searle (1983). "On the History of the Kronecker Product," *Linear and Multilinear Algebra 14*, 113–120.

Shayle Searle, Professor Emeritus, Cornell University (right)

Scandal!

H.V. Henderson, F. Pukelsheim, and S.R. Searle (1983). "On the History of the Kronecker Product," *Linear and Multilinear Algebra 14*, 113–120.

Abstract

History reveals that what is today called the Kronecker product should be called the Zehfuss Product.

This fact is somewhat appreciated by the modern (numerical) linear algebra community:

R.J. Horn and C.R. Johnson(1991). *Topics in Matrix Analysis*, Cambridge University Press, NY, p. 254.

A.N. Langville and W.J. Stewart (2004). "The Kronecker product and stochastic automata networks," *J. Computational and Applied Mathematics 167*, 429–447.

Let's Get to the Bottom of This!

**History Reveals That
What is Today Called
The Kronecker Product
Should Be Called
The Zehfuss Product**

Who Was Zeyfuss?

Born 1832.

Obscure professor of mathematics at University of Heidelberg for a while. Then went on to other things.

Wrote papers on determinants...

G. Zeyfuss (1858). "Uber eine gewasse Determinante," *Zeitscrift fur Mathematik und Physik 3*, 298–301.

Main Result a.k.a. "The Z Theorem"

If $B \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{n \times n}$ then

$$\det(B \otimes C) \;=\; \det(B)^n \det(C)^m$$

Proof: Noting that $I_n \otimes B$ and $I_m \otimes C$ are block diagonal, take determinants in

$$B \otimes C \;=\; (B \otimes I_n)(I_m \otimes C) \;=\; P(I_n \otimes B)P^T(I_m \otimes C)$$

where P is a perfect shuffle permutation.

Zehfuss(1858)

$$\Delta = \begin{vmatrix} a_1\mathcal{A}_1 & a_1\mathcal{B}_1 & b_1\mathcal{A}_1 & b_1\mathcal{B}_1 & c_1\mathcal{A}_1 & c_1\mathcal{B}_1 & d_1\mathcal{A}_1 & d_1\mathcal{B}_1 \\ a_1\mathcal{A}_2 & a_1\mathcal{B}_2 & b_1\mathcal{A}_2 & b_1\mathcal{B}_2 & c_1\mathcal{A}_2 & c_1\mathcal{B}_2 & d_1\mathcal{A}_2 & d_1\mathcal{B}_2 \\ a_2\mathcal{A}_1 & a_2\mathcal{B}_1 & b_2\mathcal{A}_1 & b_2\mathcal{B}_1 & c_2\mathcal{A}_1 & c_2\mathcal{B}_1 & d_2\mathcal{A}_1 & d_2\mathcal{B}_1 \\ a_2\mathcal{A}_2 & a_2\mathcal{B}_2 & b_2\mathcal{A}_2 & b_2\mathcal{B}_2 & c_2\mathcal{A}_2 & c_2\mathcal{B}_2 & d_2\mathcal{A}_2 & d_2\mathcal{B}_2 \\ a_3\mathcal{A}_1 & a_3\mathcal{B}_1 & b_3\mathcal{A}_1 & b_3\mathcal{B}_1 & c_3\mathcal{A}_1 & c_3\mathcal{B}_1 & d_3\mathcal{A}_1 & d_3\mathcal{B}_1 \\ a_3\mathcal{A}_2 & a_3\mathcal{B}_2 & b_3\mathcal{A}_2 & b_3\mathcal{B}_2 & c_3\mathcal{A}_2 & c_3\mathcal{B}_2 & d_3\mathcal{A}_2 & d_3\mathcal{B}_2 \\ a_4\mathcal{A}_1 & a_4\mathcal{B}_1 & b_4\mathcal{A}_1 & b_4\mathcal{B}_1 & c_4\mathcal{A}_1 & c_4\mathcal{B}_1 & d_4\mathcal{A}_1 & d_4\mathcal{B}_1 \\ a_4\mathcal{A}_2 & a_4\mathcal{B}_2 & b_4\mathcal{A}_2 & b_4\mathcal{B}_2 & c_4\mathcal{A}_2 & c_4\mathcal{B}_2 & d_4\mathcal{A}_2 & d_4\mathcal{B}_2 \end{vmatrix}$$

Zehfuss(1858)

$$p = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \qquad \text{und} \qquad P = \begin{vmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{vmatrix}$$

$$\Delta_{2\;2,2} = p_4^2 P_2^4$$

$$\Delta_{2\;Mm} = p^M P^m$$

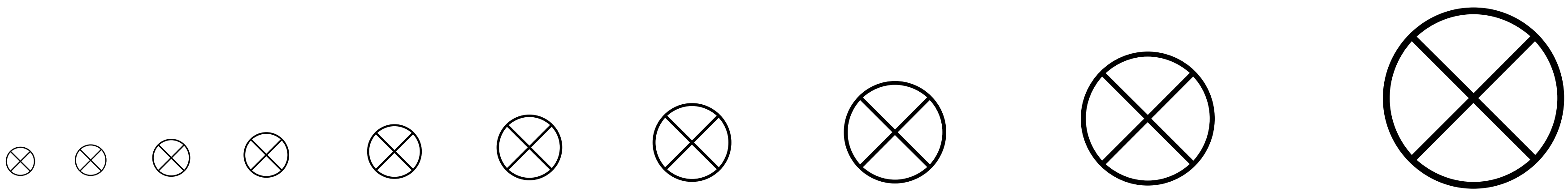
Hensel (1891)

Student in Berlin 1880-1884.

Maintains that Kronecker presented the Z-theorem in his lectures.

K. Hensel (1891). Uber die Darstellung der Determinante eines Systems, welsches aus zwei a irt ist," *ACTA Mathematica 14*, 317–319.

The 1900's



Muir (1911)

Attributes the Z-theorem to Zehfuss.

Calls $\det(B \otimes C)$ the "Zehfuss determinant."

T. Muir (1911). *The Theory of Determinants in the Historical Order of Development, Vols 1-4*, Dover, NY.

Rutherford(1933)

"On the Condition that Two Zehfuss Matrices are Equal"

$$B \otimes C \overset{???}{=} F \otimes G$$

If B $(m_b \times n_b)$, C $(m_c \times n_c)$, F $(m_f \times n_f)$, G $(m_g \times n_g)$, then $(B \otimes C)_{ij} \; = \; (F \otimes G)_{ij}$ means

$$B(\text{floor}(i/m_c), \text{floor}(j/n_c)) \cdot C(i \bmod m_c, j \bmod n_c)$$
$$=$$
$$F(\text{floor}(i/m_g), \text{floor}(j/n_g)) \cdot G(i \bmod m_g, j \bmod n_g)$$

Van Loan(1934)

"On the Condition that Two Square Zehfuss Matrices are Equal"

$$B \otimes C \overset{???}{=} F \otimes G$$

If

$$U_b^T B V_b = \mathrm{diag}(\sigma_i(B))$$

$$U_c^T C V_c = \mathrm{diag}(\sigma_i(C))$$

then

$$U_b^T F V_b = \alpha \cdot \mathrm{diag}(\sigma_i(F))$$

$$U_c^T G V_c = \frac{1}{\alpha} \cdot \mathrm{diag}(\sigma_i(G))$$

Heightened Profile Beginning in the 60s

Some Reasons:

Regular Grids

Tensoring Low Dimension Ideas

Higher Order Statistics

Fast Transforms

Preconditioners

Numerical Multi-linear Algebra

Regular Grids

$(M+1)$-by-$(N+1)$ discretization of the Laplacian on a rectangle...

$$A = I_M \otimes T_N + T_M \otimes I_N$$

$$T_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Tensoring Low Dimension Ideas

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i\, f(x_i) \;=\; w^T f(x)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,dx\,dy\,dz \approx \sum_{i=1}^{n_x}\sum_{j=1}^{n_y}\sum_{k=1}^{n_z} w_i^{(x)} w_j^{(y)} w_k^{(z)} f(x_i, y_j, z_k)$$

$$= (w^{(x)} \otimes w^{(y)} \otimes w^{(z)})^T f(x \otimes y \otimes z)$$

Higher Order Statistics

$$\mathrm{E}(xx^T)$$
$$\Downarrow$$
$$\mathrm{E}(x \otimes x)$$
$$\Downarrow$$
$$\mathrm{E}(x \otimes x \otimes \cdots \otimes x)$$

Kronecker powers:

$$\otimes^k A = A \otimes A \otimes \cdots \otimes A \qquad (k \text{ times})$$

Fast Transforms

FFT

$$F_{16}P_{16} \;=\; B_{16}(I_2 \otimes \; B_8)(I_4 \otimes B_4)(I_8 \otimes B_2)$$

$$B_4 \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega_4 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega_4 \end{bmatrix} \qquad \omega_n \;=\; \exp(-2\pi i/n)$$

Haar Wavelet Transform

$$W_{2m} \;=\; \begin{cases} \left[W_m \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| I_m \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] & \text{if } m > 1 \\ [\,1\,] & \text{if } m = 1 \end{cases} .$$

Preconditioners

If $A \approx B \otimes C$, then $B \otimes C$ has potential as a preconditioner.

It captures the essence of A.

It is easy to solve $(B \otimes C)z = r$.

Best Example: A band block Toeplitz with banded Toeplitz blocks.
B and C chosen to be band Toeplitz.

Nagy, Kamm, Kilmer, Perrone, others

Tensor Decompositions/Approximation

Given $\mathcal{A} = \mathcal{A}(1{:}n, 1{:}n, 1{:}n, 1{:}n)$, find orthogonal

$$Q = \left[\, q_1 \;\cdots\; q_n \,\right]$$

$$U = \left[\, u_1 \;\cdots\; u_n \,\right]$$

$$V = \left[\, v_1 \;\cdots\; v_n \,\right]$$

$$W = \left[\, w_1 \;\cdots\; w_n \right]$$

and a "core tensor" σ so

$$\text{vec}(\mathcal{A}) \;\approx\; \sum_{i,j,k,\ell=1}^{n} \sigma_{ijk,\ell}\, w_i \otimes v_j \otimes u_k \otimes q_\ell$$

Offspring

1. The Left Kronecker Product
2. The Hadamard Product
3. The Tracy-Singh Product
4. The Khatri-Rao Product
5. The Generalized Kronecker Product
6. The Symmetric Kronecker Product
7. The Bi-Alternate Product

Left Kronecker Product

Definition:

$$B \overset{\text{Left}}{\otimes} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11}B & c_{12}B \\ c_{21}B & c_{22}B \end{bmatrix} = C \otimes B$$

Fact:

If $B \in \mathbb{R}^{m_b \times n_b}$ and $C \in \mathbb{R}^{m_c \times n_c}$ then

$$B \overset{\text{Left}}{\otimes} C = \Pi^T_{m_c,\, m_b m_c} (B \otimes C) \Pi_{n_c,\, n_b n_c}$$

$\uparrow$ Perfect Shuffles $\uparrow$

The Hadamard Product

Definition:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \overset{\text{Had}}{\otimes} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} b_{11}c_{11} & b_{12}c_{12} \\ b_{21}c_{21} & b_{22}c_{22} \\ b_{31}c_{31} & b_{32}c_{32} \end{bmatrix}$$

$$B \overset{\text{Had}}{\otimes} C \;=\; B .* C$$

The Hadamard Product

Fact:

If $\tilde{A} = B \otimes C$ and $B, C \in {\rm I\!R}^{m \times n}$, then

$$B \overset{\text{Had}}{\otimes} C = \tilde{A}(1{:}(m+1){:}m^2, 1{:}(n+1){:}n^2)$$

E.g.,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \otimes \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \left[\begin{array}{cc|cc} \color{red}{b_{11}c_{11}} & b_{11}c_{12} & b_{12}c_{11} & \color{red}{b_{12}c_{12}} \\ b_{11}c_{21} & b_{11}c_{22} & b_{12}c_{21} & b_{12}c_{22} \\ b_{11}c_{31} & b_{11}c_{32} & b_{12}c_{31} & b_{12}c_{32} \\ \hline b_{21}c_{11} & b_{21}c_{12} & b_{22}c_{11} & b_{22}c_{12} \\ \color{red}{b_{21}c_{21}} & b_{21}c_{22} & b_{22}c_{21} & \color{red}{b_{22}c_{22}} \\ b_{21}c_{31} & b_{21}c_{32} & b_{22}c_{31} & b_{22}c_{32} \\ \hline b_{31}c_{11} & b_{31}c_{12} & b_{32}c_{11} & b_{32}c_{12} \\ b_{31}c_{21} & b_{31}c_{22} & b_{32}c_{21} & b_{32}c_{22} \\ \color{red}{b_{31}c_{31}} & b_{31}c_{32} & b_{32}c_{31} & \color{red}{b_{32}c_{32}} \end{array}\right]$$

The Tracy-Singh Product

Definition:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$B \overset{\text{TS}}{\otimes} C = \left[\begin{array}{cc|cc} B_{11} \otimes C_{11} & B_{11} \otimes C_{12} & B_{12} \otimes C_{11} & B_{12} \otimes C_{12} \\ B_{11} \otimes C_{21} & B_{11} \otimes C_{22} & B_{12} \otimes C_{21} & B_{12} \otimes C_{22} \\ \hline B_{21} \otimes C_{11} & B_{21} \otimes C_{12} & B_{22} \otimes C_{11} & B_{22} \otimes C_{12} \\ B_{21} \otimes C_{21} & B_{21} \otimes C_{22} & B_{22} \otimes C_{21} & B_{22} \otimes C_{22} \\ \hline B_{31} \otimes C_{11} & B_{31} \otimes C_{12} & B_{32} \otimes C_{11} & B_{32} \otimes C_{12} \\ B_{31} \otimes C_{21} & B_{31} \otimes C_{22} & B_{32} \otimes C_{21} & B_{32} \otimes C_{22} \end{array}\right]$$

The Khatri-Rao Product

Definition:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}$$

$$B \overset{\text{K-R}}{\otimes} C = \left[\begin{array}{c|c} B_{11} \otimes C_{11} & B_{12} \otimes C_{12} \\ \hline B_{21} \otimes C_{21} & B_{22} \otimes C_{22} \\ \hline B_{31} \otimes C_{31} & B_{32} \otimes C_{32} \end{array}\right]$$

The Khatri-Rao Product

Fact:

$$B \overset{\text{KR}}{\otimes} C \quad \text{is a submatrix of} \quad B \overset{\text{TS}}{\otimes} C$$

$$B \overset{\text{TS}}{\otimes} C = \left[\begin{array}{cc|cc} B_{11}\otimes C_{11} & B_{11}\otimes C_{12} & B_{12}\otimes C_{11} & B_{12}\otimes C_{12} \\ B_{11}\otimes C_{21} & B_{11}\otimes C_{22} & B_{12}\otimes C_{21} & B_{12}\otimes C_{22} \\ B_{11}\otimes C_{31} & B_{11}\otimes C_{32} & B_{12}\otimes C_{31} & B_{12}\otimes C_{32} \\ \hline B_{21}\otimes C_{11} & B_{21}\otimes C_{12} & B_{22}\otimes C_{11} & B_{22}\otimes C_{12} \\ B_{21}\otimes C_{21} & B_{21}\otimes C_{22} & B_{22}\otimes C_{21} & B_{22}\otimes C_{22} \\ B_{21}\otimes C_{31} & B_{21}\otimes C_{32} & B_{22}\otimes C_{31} & B_{22}\otimes C_{32} \\ \hline B_{31}\otimes C_{11} & B_{31}\otimes C_{12} & B_{32}\otimes C_{11} & B_{32}\otimes C_{12} \\ B_{31}\otimes C_{21} & B_{31}\otimes C_{22} & B_{32}\otimes C_{21} & B_{32}\otimes C_{22} \\ B_{31}\otimes C_{31} & B_{31}\otimes C_{32} & B_{32}\otimes C_{31} & B_{32}\otimes C_{32} \end{array}\right]$$

The Generalized Kronecker Product

Regalia and Mitra (1989), "Kronecker Products, Unitary Matrices, and Signal Processing Applications," *SIAM Review, 31*, 586–613.

$$\left\{ \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \right\} \overset{\text{gen}}{\otimes} C \;=\; \left[\begin{array}{c} B_1 \overset{\text{Left}}{\otimes} C(1,:) \\ \\ B_2 \overset{\text{Left}}{\otimes} C(2,:) \\ \\ B_3 \overset{\text{Left}}{\otimes} C(3,:) \\ \\ B_4 \overset{\text{Left}}{\otimes} C(4,:)] \end{array} \right]$$

The Generalized Kronecker Product

$$\left\{\begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array}\right\} \overset{\text{GEN}}{\otimes} \left\{\begin{array}{c} C_1 \\ C_2 \end{array}\right\} = \left\{\begin{array}{c} \left\{\begin{array}{c} B_1 \\ B_2 \end{array}\right\} \overset{\text{gen}}{\otimes} C_1 \\ \\ \left\{\begin{array}{c} B_3 \\ B_4 \end{array}\right\} \overset{\text{gen}}{\otimes} C_2 \end{array}\right\}$$

The Symmetric Kronecker Product

Kronecker Product turns matrix equations into vector equations:

$$CXB^T = G \qquad \Leftrightarrow \qquad (B \otimes C)\ \mathrm{vec}(X) = \mathrm{vec}(G)$$

The symmetric Kronecker product does the same thing for matrix equations with symmetric solutions:

$$\tfrac{1}{2}\left(CXB^T + BXC^T\right) = G \qquad \text{(symmetric)}$$

$$\Leftrightarrow$$

$$(B \overset{\text{sym}}{\otimes} C)\ \mathrm{svec}(X) = \mathrm{svec}(G)$$

where

$$\mathrm{svec}\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}\right) = \begin{bmatrix} x_{11} & \sqrt{2}x_{12} & x_{22} & \sqrt{2}x_{13} & \sqrt{2}x_{23} & x_{33} \end{bmatrix}^T$$

Symmetric Kronecker Product

Fact:
If

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & \alpha & 0 & 0 & 1 \end{bmatrix} \qquad \alpha = 1/\sqrt{2}$$

then $\mathrm{vec}(X) = P \cdot \mathrm{svec}(X)$ and

$$B \overset{\text{sym}}{\otimes} C = P^T (B \otimes C) P$$

Bi-Alternate Product

$$B \otimes C \; = \; \frac{1}{2}(B \otimes C \; + \; C \otimes B)$$

W. Govaerts, *Numerical Methods for Bifurcations of Dynamical Equilibria*, SIAM.

The 2000's

Three Predictions

Big N Will Mean Big d Will Mean KP

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

$$N = 2^d$$

Think Scalar Think Block Think Tensor

If for all $1 \leq m_i \leq n$ we have,

$$\mathcal{B}(m_1, m_2, m_3, m_4)$$
$$=$$
$$\sum_{i_1,i_2,i_3,i_4=1}^{n} W(i_1, m_1)Y(i_2, m_2)X(i_3, m_3)Z(i_4, m_4)\mathcal{A}(i_1, i_2, i_3, i_4)$$

then

$$B \;=\; (W \otimes Y)^T A(X \otimes Z)$$

Data-Sparse Approximate Factorizations

$$A \;\approx\; (B_1 \otimes C_1)(B_2 \otimes C_2)(B_3 \otimes C_3)\cdots$$

Det(Log(A))

$$A \approx (B \otimes C)(D \otimes E)(F \otimes G)\cdots$$

$$\log(\det(A)) \approx \begin{array}{l} n_c \log(\det(B)) + n_b \log(\det(C)) + \\ n_e \log(\det(D)) + n_d \log(\det(E)) + \\ n_g \log(\det(F)) + n_f \log(\det(G)) \cdots \end{array}$$

See Zehfuss (2058).