From *SIAM News*, Volume 40, Number 8, October 2007

Parabolic PDEs and Deterministic Games

By Robert V. Kohn

Parabolic partial differential equations seem very different from first-order Hamilton–Jacobi equations. Parabolic equations are linked to random walks, and often arise as steepest descents; Hamilton–Jacobi equations have characteristics, and often arise from optimal control problems.

But appearances can be deceiving. Some parabolic (and elliptic) PDEs are surprisingly similar to Hamilton–Jacobi equations. In my talk at ICIAM, I explained why, drawing on recent work with Sylvia Serfaty [7].

To keep things simple, I focus on two key examples:

(i) *Motion with constant velocity.* Consider the evolution of a region Ω in the plane as its boundary moves inward with constant velocity 1 (Figure 1, left). The evolution is completely characterized by the arrival time

$$u(x) = \text{time at which the moving boundary passes through } x.$$

This function solves the Hamilton–Jacobi equation

$$|\nabla u| = 1 \text{ in } \Omega \tag{1}$$

with $u = 0$ at the boundary, and it is characterized by the optimization

$$u(x) = \min_{z \in \partial\Omega} \operatorname{dist}(x, z). \tag{2}$$

(ii) *Motion by curvature*. Now consider the evolution of a convex region Ω in the plane as its boundary moves with velocity equal to its curvature (Figure 1, right). To track the evolution of the boundary as a parameterized curve, we must solve a nonlinear parabolic PDE. But if the region is initially convex, it stays convex, so the evolution is again completely characterized by the arrival time u. A moment's thought reveals that $-\operatorname{div}(\nabla u/|\nabla u|)$ is the curvature of a level set of u, and the velocity of the moving front is $1/|\nabla u|$, so the arrival time of motion by curvature solves

$$|\nabla u| \operatorname{div}(\nabla u / |\nabla u|) + 1 = 0 \text{ in } \Omega \tag{3}$$

with $u = 0$ at the boundary. This PDE is to motion by curvature as the eikonal equation (1) is to motion with constant velocity.

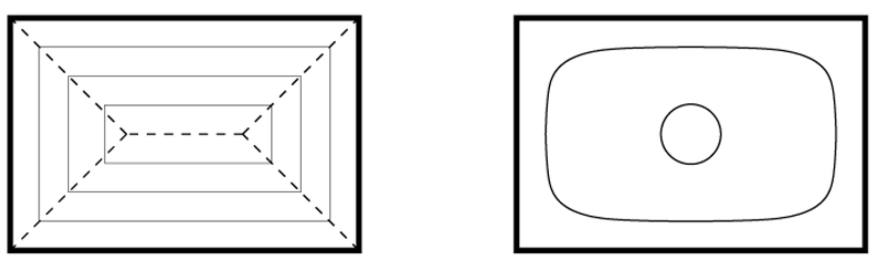

Figure 1. *Left, Motion with constant velocity. Right, Motion by curvature.*

I claim that these evolutions are similar, in the sense that *motion by curvature also has a deterministic control interpretation*, analogous to (2). It involves a two-person game, with players Paul and Carol and a small parameter ε. Paul is initially at some point $x \in \Omega$; his goal is to exit as soon as possible. Carol wants to delay his exit as long as possible. The game proceeds as follows:

- Paul chooses a direction, i.e., a unit vector $|v| = 1$.
- Carol can either accept or reverse Paul's choice, i.e., she chooses $b = \pm 1$.
- Paul then moves distance $\sqrt{2}\varepsilon$ in the possibly reversed direction, i.e., from x to $x + \sqrt{2}\varepsilon bv$.
- This cycle repeats until Paul reaches $\partial\Omega$.

Consider Paul's choice at a point near the top of the rectangle. One might think he should choose v pointing north. But that's a bad idea: If he does so, Carol will reverse him and he'll have to go south (Figure 2, left).

Can Paul exit? Yes indeed. This is easiest to see when $\partial\Omega$ is a circle of radius R. The midpoints of secants of length $2\sqrt{2}\varepsilon$ trace a concentric circle, whose radius is smaller than R by approximately ε^2/R. Paul can exit in one step if and only if he starts on or outside this concentric circle (Figure 2, middle). This construction can be repeated, of course, producing a sequence of circles from which he can exit in a fixed number of steps (Figure 2, right). Aside from the scale factor of ε^2, the circles are shrinking with normal velocity $1/R$ = curvature. We have determined Paul's optimal strategy: If $\Omega = B_R(0)$ and his present position is x, then his optimal v is perpendicular to the line joining x to 0. And we have linked his minimum exit time to motion by curvature.

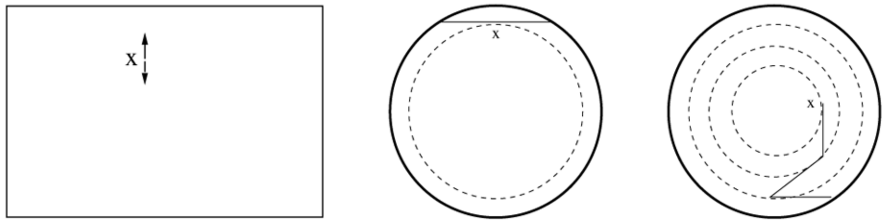

Figure 2. *Left, Paul's quandary—if he tries to go north, Carol will send him south. Middle, Paul can exit from a well-chosen concentric circle in just one step. Right, The construction can be repeated.*

This calculation is fundamentally local, so it is not really limited to circles. It suggests that Paul's scaled arrival time,

$$u_\varepsilon(x) = \varepsilon^2 \cdot \left\{ \begin{array}{l} \text{minimum number of steps} \\ \text{Paul needs to exit starting} \\ \text{from } x\text{, assuming that} \\ \text{Carol behaves optimally} \end{array} \right\}, \tag{4}$$

converges as $\varepsilon \to 0$ to the arrival-time function of motion by curvature. It even provides us with something resembling characteristics for the second-order PDE (3). In fact: Paul's paths are like characteristics, in the sense that the PDE becomes an ODE when restricted to the path (u_ε decreases by exactly ε^2 at each step along Paul's path).

The circle is too easy. How can we analyze more general domains? A key tool is the *dynamic programming principle*:

$$u_\varepsilon(x) = \min_{|v|=1} \max_{b=\pm 1} \left\{ u_\varepsilon\left(x + \sqrt{2}\varepsilon b v\right) + \varepsilon^2 \right\}. \tag{5}$$

In words: Starting from x, Paul selects the best direction v (taking into account that Carol is working against him), recognizing that after taking *this* step he will pursue an optimal path. This principle captures the logic we used in passing from the middle to the right-hand frame of Figure 2.

The degenerate-elliptic equation (3) is, in essence, the Hamilton–Jacobi–Bellman equation associated with this dynamic programming principle. To explain why, we use an argument that's familiar from optimal control theory (see, for example, Chapter 10 of [4]). Assume that u_ε is smooth enough for Taylor expansion to be valid. Then (5) gives

$$u_\varepsilon(x) \approx \min_{|v|=1} \max_{b=\pm 1} \left\{ \begin{array}{l} u_\varepsilon(x) + \sqrt{2}\varepsilon b v \cdot \nabla u_\varepsilon(x) \\ +\varepsilon^2 \left\langle D^2 u_\varepsilon(x) v, v \right\rangle + \varepsilon^2 \end{array} \right\},$$

whence

$$0 \approx \min_{|v|=1} \max_{b=\pm 1} \left\{ \begin{array}{l} \sqrt{2}\varepsilon b v \cdot \nabla u_\varepsilon(x) \\ +\varepsilon^2 \left\langle D^2 u_\varepsilon(x) v, v \right\rangle + \varepsilon^2 \end{array} \right\}.$$

Paul should choose v such that $v \cdot \nabla u_\varepsilon(x) = 0$; otherwise, this term will dominate the righthand side and Carol will choose the sign of b to make it positive. In the plane there are two such vectors v; the choice isn't important, because the next term is quadratic. We conclude (formally, in the limit $\varepsilon \to 0$) that

$$\left\langle D^2 u \frac{\nabla u^\perp}{|\nabla u|}, \frac{\nabla u^\perp}{|\nabla u|} \right\rangle + 1 = 0.$$

A bit of manipulation reveals that this is the same as (3) in two space dimensions.

To summarize: Motion by curvature is similar to motion with constant velocity in the sense that both evolutions can be described by deterministic control problems (the Paul–Carol game versus equation (2)). The PDE that describes the arrival time is, in either case, the associated Hamilton–Jacobi–Bellman equation, derived from the control problem with the principle of dynamic programming. There is, however, a difference: The Paul–Carol game has a small parameter ε, and we get motion by curvature only in the limit $\varepsilon \to 0$; the optimal control interpretation of the eikonal equation, by contrast, has no small parameter.

My discussion has been formal, and I have discussed just the simplest example. But these ideas can be justified and extended to other geometric motions. In particular:

■ The convergence of Paul's scaled arrival time u_ε to the arrival time of motion by curvature can be proved using the framework of "viscosity solutions." When u is smooth enough, an alternative is to use a "verification argument," which gives a stronger result, by estimating the convergence rate.
■ The case when Ω is nonconvex is more subtle. Then $\lim_{\varepsilon \to 0} u_\varepsilon$ is the arrival time for a different motion law, the one with normal velocity κ_+, where κ is curvature and $\kappa_+ = \max\{\kappa, 0\}$. (The proof depends on a uniqueness result for viscosity solutions, which was proved by Guy Barles and Francesca Da Lio.)
■ These ideas can be extended to higher space dimensions and other geometric evolutions. Moreover, the method can be used for parabolic as well as elliptic representations of curvature-driven motion.

Should we be surprised to see such a close connection between motion with constant velocity and motion by curvature? Perhaps not. Twenty years ago we learned that the "level set method" provides a convenient numerical approach to both problems [8]. A few years later we learned that the theory of viscosity solutions, invented with optimal control in mind, was a good analytical tool for proving theorems about motion by curvature [2,5]. In retrospect, these links were indications of a deeper connection.

Our "game" is basically a semi-discrete approximation scheme (continuous in space, discrete in time) for motion by curvature. Similar semi-discrete schemes have been considered in the literature on computer vision (e.g., [1,6,9]), and in work on numerical schemes for computing viscosity solutions of second-order PDEs [3].

The extensions mentioned above all involve *geometric* evolution problems. But is this idea limited to geometric problems? Or might other second-order elliptic and parabolic problems have optimal control interpretations? For example, does the linear heat equation have a deterministic-game interpretation?

The answer, surprisingly, is yes: There is a deterministic-game interpretation of the linear heat equation. One approach (suggested to us by H. Mete Soner) is closely related to the Black–Scholes theory of option pricing.

Even more surprising: A broad class of fully nonlinear parabolic and elliptic PDEs have deterministic interpretations. This is the focus of current work with Sylvia Serfaty.

By the way, we didn't invent the Paul–Carol game. It was introduced thirty years ago by Joel Spencer, as a heuristic for the study of certain combinatorial problems [10].

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