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Model Reduction at ICIAM ’07

By Peter Benner

In recent years, model reduction has become an ubiquitous tool in a variety of application areas and, accordingly, a research focus for many mathematicians and engineers. This was reflected in the strong representation of the topic at ICIAM ’07. Among the many application areas addressed were the analysis and simulation of dynamical systems, control design, circuit simulation, micro- and nano-electromechanical systems, structural dynamics, and computational fluid dynamics. In some of these areas, model reduction has become indispensable: An example is the design of next-generation ICs, like CPUs or memory chips, where the ever-increasing package density forces developers to include more and more (side-)effects. These devices are often modeled by very large RLC circuits; including them all in a circuit model would be far too demanding computationally—hence the need for reduced-order models to replace the originals in the full simulation runs.

Techniques that reduce the complexity of a mathematical model have been developed under several names: The most common, besides model(-order) reduction, are dimension reduction, system reduction, and order reduction. Unfortunately, some of these expressions are also used to describe techniques or phenomena quite different from the algorithmic development that was discussed in more than thirty talks during ICIAM ’07 and that is the topic of this article. The term “model reduction” is also used to describe the actual reduction of a complex physical model to a less computationally expensive one—through, for example, elimination of a space dimension in a partial differential equation model by taking ad-vantage of symmetry or other structural properties, or through neglect of time-dependencies if only steady-state behavior is of interest. “Order reduction” is a phenomenon encountered in, say, the application of integrators for ordinary differential equations to differential–algebraic equations—the order of accuracy drops in moving from ODEs to DAEs. We are also not concerned here with “dimension reduction,” as our main focus is *control systems* of the form

$$\Sigma : \begin{cases} E\dot{x}(t) = f(t, x(t), u(t)), \\ \;\; y(t) = g(t, x(t), u(t)), \end{cases} \tag{1}$$

where f, g are sufficiently smooth functions, $x(t) \in \mathbb{R}^n$ is the state of the system, and u and y play the roles of inputs (e.g., controls) and outputs (e.g., measurements), respectively. (If the $n \times n$ matrix E is singular—that is, if the dynamics of the system are given by a DAE—one also speaks of a *descriptor system*.) “Dimension reduction” is usually associated with *free* dynamical systems

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \tag{2}$$

and refers to analytical or numerical techniques for solving the ODE (2) in a lower-dimensional space by, say, modal analysis/truncation (a particularly successful technique for linear systems in structural dynamics) or projection onto center and (approximate) inertial manifolds (used mostly for nonlinear systems). These dimension-reduction techniques are less successful when applied to control systems of the form (1), for which system-theoretic properties like input–output behavior are the most interesting features. Techniques focusing on this aspect might be better called “system-reduction” methods, but we use the more common “model reduction,” mainly as a synonym for computational techniques for replacing (1) by a system of reduced-order,

$$\widehat{\Sigma} : \begin{cases} \widehat{E}\dot{\hat{x}}(t) = \hat{f}(t, \hat{x}(t), u(t)), \\ \;\; \hat{y}(t) = \hat{g}(t, \hat{x}(t), u(t)), \end{cases} \tag{3}$$

where the state $\hat{x}$ now lives in $\mathbb{R}^r$, $r \ll n$. The goal of model reduction is to find $\widehat{\Sigma}$ so that $\| y - \hat{y} \|$ is small in some appropriate measure if the same input function u is used to drive the systems Σ and $\widehat{\Sigma}$.

Numerous speakers at ICIAM ’07 highlighted the latest developments and achievements in model reduction and applications thereof. The majority of these speakers considered linear systems, i.e., $f(t, x, u) = Ax + Bu$, $g(t, x, u) = Cx + Du$ in (1), with matrices A, B, C, D of appropriate dimensions, where analogous linear reduced-order models are sought. An often used tool for describing input–output behavior of linear systems is the transfer function

$$G(s) = C(sE - A)^{-1} B + D$$

(with $\widehat{G}$ denoting the transfer function of the reduced-order system) that results from the Laplace transform of the linear system. Model reduction for linear systems, then, can also be posed as an approximation problem, min $\| G - \widehat{G} \|$ in some appropriate norm (usually the Hardy H_2- or H_∞-norms or the Hankel norm).

Even with the advanced theory and methodology that have been developed for linear systems, many open problems and challenges remain. As to nonlinear systems, it became clear in several talks that the general application of model-reduction methods devised for nonlinear systems will be a major challenge for future research.

Model reduction featured most prominently in Zurich in a plenary talk, “Model reduction of DAEs,” by Tatjana Stykel (TU Berlin). In her talk, one of two Richard von Mises Prize Lectures* given in Zurich, Stykel described the computation of reduced-order models for descriptor systems via balanced truncation. This method and its offspring are known to be efficient and reliable techniques for linear ODEs, providing computable error bounds that make it possible to adapt the reduced-order r to a prescribed error threshold and preserve important physical properties in the reduced-order model. Sty-kel's work is particularly important in that it allows the application of these techniques to the linear circuit models mentioned earlier (which usually take the form of DAEs) and also has considerable potential for the control of mechanical (multibody) and electromechanical systems (robots, MEMS, . . .)—a natural source of large-scale DAE systems.

Tatjana Stykel (TU Berlin) gave a Richard von Mises Prize Lecture in Zurich at the embedded GAMM meeting. Stykel, shown here with Volker Mehrmann and GAMM president Rolf Jeltsch, titled her lecture "Model Reduction of Differential–Algebraic Equations."

A wide range of methods, theoretical results, and application areas were covered in three two-part minisymposia, with four talks each, organized by Volker Mehrmann (TU Berlin), and Zhaojun Bai and Roland Freund (of UC Davis). These minisymposia ran consecutively, Monday through Thursday, and almost had the character of an embedded meeting. The first minisymposium, "Theory, Methodology and Software," comprised a series of talks by Peter Benner and Ulrike Baur (Chemnitz University of Technology), Dan Sorensen (Rice University), and Ming Gu (UC Berkeley) on developments in balanced truncation and the related topic of matrix equations; these speakers demonstrated that with modern techniques from numerical linear algebra, these methods can be applied efficiently to systems of order up to $n = 10^6$. A second set of talks then highlighted interpolation methods; these methods prescribe interpolation conditions for the transfer function $\widehat{G}$ of the reduced-order system of the form

$$\frac{d^k}{ds^k}\widehat{G}(s_j) = \frac{d^k}{ds^k}G(s_j), j = 0,\ldots,\ell, \quad k = 0,\ldots,q_j,$$

and thus generalize the well-known moment-matching Padé or Padé-like methods for which $\ell = 0$. Of particular interest here is the question, only partially answered, of where the interpolation points should be placed and how many derivatives need to be matched for the best approximation properties. Thanos Antoulas (Rice University), and Serkan Gugercin and Christopher Beattie (Virginia Tech) described significant progress in this direction.

The focus in the second minisymposium, "Structured and Higher-order Systems," was on methods for addressing special structural properties, such as passivity and dependence on parameters (other than time or frequency). David Bindel (Courant Institute), Roland Freund, Lihong Feng (Chemnitz University of Technology), Wen-Wei Lin (National Tsing Hua University, Taiwan), Mark Embree (Rice University), Zhaojun Bai, Jaijeet Roychowdhury (University of Minnesota), and Patricia Astrid (Shell B.V.) considered several diverse aspects, from MEMS applications to highly oscillatory systems and damped wave equations.

The third and most application-oriented minisymposium in the series was titled "Model Reduction in Circuit Simulation"; appropriately, four of the seven speakers were from industry. Jan ter Maten and Wil Schilders (of NXP Semiconductors, Eindhoven), Luca Daniel (MIT), Yangfeng Su (Fudan University, Shanghai), Stefan Reitzinger (Computer Simulation Technology, Darmstadt), Tatjana Stykel, and Peter Feldmann (IBM T.J. Watson Research Center) focused on applications in microelectronics—like the problems in circuit simulation mentioned earlier, e.g., interconnect modeling or passivity preservation in the computation of reduced-order models for passive devices. New challenges are posed by systems with huge numbers of inputs and outputs, like those describing on-chip clock-distribution networks, on-chip power grids, or wide buses. Feldmann described a recursive singular value decomposition technique for treating such systems. The principal idea is based on the observation that transfer functions of such systems often have a block structure, with low-rank blocks. The initial results are promising, but this new challenge in linear model reduction calls for further research.

Complementing the talks mentioned here were others scattered throughout the program that touched on model reduction in one way or another. (A search of the book of abstracts yields more than 60 hits for "model reduction," and other search terms would probably elicit many more.)

*The Richard von Mises Prize is awarded annually by the *Gesellschaft für Angewandte Mathematik und Mechanik* (GAMM) to a young (no older than 36) researcher for outstanding scientific contributions in applied mathematics and mechanics. The two Von Mises prize recipients who spoke in Zurich were Stykel and Michael Dumbser (Universität Stuttgart).

For instance, speakers in the minisymposium “Numerical Methods for Data-assimilation Problems,” co-organized by Angelika Bunse-Gerstner (Bremen University) and Nancy Nichols (University of Reading), discussed the application of model reduction for state-estimation problems in large-scale systems. Given the sheer number of interesting lectures, often overlapping (which is of course unavoidable at a congress the size of this year’s ICIAM), no one person could possibly have attended all the talks on model reduction. But the large number of talks related to model reduction also shows that it has become an important and integral part of industrial and applied mathematics!

Altogether, ICIAM ’07 provided an excellent forum and stimulating atmosphere for research on model reduction, not only for people now working in the area, but also for those who might be interested in applying novel techniques or who are just starting to work in the area. Anyone who attended the majority of the minisymposium talks and Stykel’s plenary lecture should have come away with a good understanding of the current state of the area and the future challenges. I believe that the stimulating atmosphere will lead to continued activity, maybe even to breakthroughs for some of the problems discussed in Zurich!

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