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Manipulating Electromagnetic Fields: Mathematics, Metamaterials, and Cloaking

Intrigued by Bob Kohn's invited talk at the 2008 SIAM Conference on Mathematical Methods for Materials Science (May 11–13), SIAM technical director Bill Kolata put together the following overview for readers of SIAM News.

The interaction of physical theory, mathematics, and engineering design to manipulate electromagnetic fields for practical ends has a long and rich history. As far back as the 10th century, the Basrah (Iraq)-born scientist and mathematician Al-Haitham (Al-hazen) wrote a treatise on optics, *Kitab-al-Manazir*. Translated into Latin (*Opticae Thesaurus*), Alhazen's work formed the basis for the development of the first spectacles (reading stones) by monks in Europe. Approximately 900 years later, in 1893, J.J. Thomson proposed the first waveguide, which was experimentally verified by O.J. Lodge a year later. In 1897, Lord Rayleigh performed the first mathematical analysis of the propagating modes within a hollow metal cylinder. A more recent example is "stealth" technology. In each of these examples, and in the many others that could be cited, the manipulation was achieved by designing and controlling the interfaces between materials.

Materials Not Seen in Nature

In the late 1990s, scientists and engineers began to think that it might be possible to build periodic microstructures and, taking advantage of their resonances, design materials with properties not observed in nature (hence the name "metamaterials"). In the late 1960s, the Soviet physicist Victor Veselago had considered materials with negative refractive index, showing that, if they existed, a "super-lens" would be possible. This super-lens would be able to focus features much smaller than the wavelength of the light illuminating the object. In 2000, John Pendry described a slab constructed of a metamaterial with a negative refractive index that was a super-lens [7]. In 2005, two independent teams of researchers showed that an extremely thin silver foil has a negative refractive index and acts as a super-lens.

At the same time, mathematical analysis of these and other new materials was providing insight into their properties and their potential for applications. Perhaps the most widely publicized of recent applications is the ability to render an object invisible to probing by electromagnetic waves by surrounding it with a "cloak," a suitably designed heterogeneous, anisotropic, and dielectric metamaterial. In 2006, Pendry, Schurig, and Smith described the design of a cloak for the time-harmonic Maxwell equations [8]. Greenleaf, Lassas, and Uhlmann had made essentially the same observation three years earlier, in the context of electric impedance tomography [2]. The idea in both cases is to take advantage of the form invariance of the equations, using a change of variables, in designing a cloak. Other approaches to cloaking have been developed, including anomalous localized resonances [6], optical conformal mapping [4], and extensions to elastodynamic equations [5]; an interesting history of cloaking can also be found in [5].

Analytically, the most realistic and difficult problem is invisibility to the scattering of pulsed electromagnetic waves. A bit easier is invisibility to fixed-frequency wave scattering, and still easier is electrostatics (zero-frequency), as in electric impedance tomography. In the latter case, $\nabla\,(\sigma\nabla u) = 0$ in the interior of a region Ω, where σ is the bounded, positive-definite conductivity tensor. The sensing mechanism is given by the Dirichlet-to-Neumann map $\Lambda_\sigma : f \to g$, which takes a voltage $f(x)$ defined on the boundary Γ and maps it to the current flux $g = (\sigma\nabla u)\cdot v$ on Γ, where v is the outward unit normal to Γ. In this context, cloaking means that for a region D in the interior of Ω we can define $\sigma_c(x)$ in $\Omega\backslash D$ such that the boundary measurements look the same as those for the uniform case $\sigma \equiv 1$, regardless of the conductivity in D.

Change-of-Variables Approach to Cloak Design

The motivation for defining the cloak by change of variables is the following. Λ_σ can be characterized by a variational principle:

$$\int_\Gamma f\cdot\Lambda_\sigma(f) = \min_{u=f}\int_\Omega \langle \sigma\nabla u, \nabla u\rangle dx.$$

Moreover, the equations including the Dirichlet-to-Neumann map are invariant under a change of variables. Thus, if $y = F(x)$ is an invertible, orientation-preserving change of variables on Ω with $F(x) = x$ on the boundary Γ, then by change of variables:

$$\int_\Omega \langle \sigma\nabla u, \nabla u\rangle dx = \int_\Omega \langle F_* \sigma\cdot\nabla w, \nabla w\rangle dy,$$

where $y = F(x)$, $u(x) = w(y)$,

$$F_*\sigma(y) = \frac{1}{\det(DF)(x)} DF(x)\sigma(x)DF(x)^T,$$

where $DF = \partial y/\partial x$,

and, because $F(x) = x$ on Γ, the boundary measurements for σ and $F_*\sigma$ are identical. Therefore, they have the same Dirichlet-to-Neumann map.

To see how this can be applied to cloaking, consider the example $\Omega = B_2$, the ball of radius 2 centered at 0, and suppose we want to cloak the region defined by the unit ball B_1, assuming that the conductivity in B_1 is given by an arbitrary function $A(y)$. We want to define the conductivity in $B_2\backslash B_1$ in such a way that B_1 is cloaked.

We start by considering a ball B_ρ of (small) radius ρ in B_2 and assume the conductivity in $B_2\backslash B_\rho$ to be 1; in other words, this is a case of uniform conductivity in a region with a small inclusion [3]. We then define a continuous, piecewise-smooth change of variables $y = F(x)$ that maps B_2 onto itself and B_ρ onto B_1, and for which $F(x) = x$ on the outer boundary. One such map is

$$F(x) = x/\rho, \text{for} |x| \leq \rho,$$

$$F(x) = \tfrac{1}{2-\rho}(2 - 2\rho + |x|)\frac{x}{|x|},$$

$$\text{for } \rho \leq |x| \leq 2.$$

Under this change of variables, in the image B_2 of F, the conductivities are $A(y)$ in B_1 and $F_*1(y)$ in $B_2\backslash B_1$. It is $F_*1(y)$ that provides cloaking for $A(y)$ in B_1. In the domain of F, the corresponding conductivities are $F_*^{-1}A(x)$ in B_ρ and 1 in $B_2\backslash B_\rho$. These two cases are therefore indistinguishable under the Dirichlet-to-Neumann map. But $F_*1(y)$ is not quite a cloak for the region B_1; rather, it is a "near-cloak," in the sense that in a natural operator norm, the difference in the Dirichlet-to-Neumann map corresponding to uniform conductivity 1 on B_2 and that corresponding to conductivity $A(y)$ in B_1 and $F_*1(y)$ in $B_2\backslash B_1$ is bounded by $C\rho^{-n}$, where n is the dimension of the space. The final step is to show that in the singular limit as $\rho \to 0$, the cloak is complete. In this case, F is given by

$$F(x) = \frac{1}{2}(2 + |x|)\frac{x}{|x|}.$$

This step is not straightforward, because the validity of change of variables is called into question by the singularity of F. Moreover, the cloak F_*1 is singular on the boundary $|x| = 1$, which requires that some care be taken in defining what is meant by a solution to the PDE. However, $F_*^{-1}A(x)$ is a single point and can be treated as a removable singularity.

New and Future Directions

Formally, change of variables can be ex-tended to the scattering of waves of an arbitrary but fixed non-zero frequency. Small inclusions are not necessarily negligible in wave propagation problems; nevertheless, a near-cloak is still possible. Recently, Kohn, Onofrei, Vogelius, and Weinstein found near-cloaking to be possible at finite frequency by a similar change-of-variables-based construction. Their argument requires a lossy layer at the edge of the cloak, however, and the quality of the resulting near-cloak is much poorer than for impedance tomography. In the case of the Helmholtz equation in two dimensions, for example, the norm of the difference of the resulting Dirichlet-to-Neumann maps is bounded by $C\,|\log\rho\,|^{-1}$ rather than $C\,|\,\rho\,|^{-2}$.

In other recent work, Greenleaf, Kurylev, Lassas, and Uhlmann took an approach based on the notion of a "finite energy solution" to the Helmholtz equation [1]. With a somewhat different (but still change-of-variables-based) "double-coating" construction, they have been able to cloak even active sources.

Substantial hurdles would have to be overcome to make cloaking practical. Current schemes, for example, require highly anisotropic dielectrics with singularities at cloak boundaries (although this may be achievable at specific frequencies with metamaterials). And while the construction of cloaks is not frequency-dependent, the dielectric properties of materials are frequency-dependent, and this poses a problem for cloaking from pulsed electromagnetic sources.

Mathematics continues to provide insights into the design of metamaterials, and advances in the understanding and construction of these novel materials continue to provide ways to implement new applications.

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