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Spinning into Posterity

By Dana Mackenzie

Of all the equations that scientists have written over the centuries, only one has been chiseled in stone and placed in Westminster Abbey. Not far from the grave of Isaac Newton, a green slate plaque, two feet by two feet, bears the following inscription:

$$i\gamma\cdot\partial\psi = m\psi.$$

The stone is a memorial for Paul Dirac (1902–1984); placed in 1995, it is an appropriate tribute to a man who spoke little but had a great love for mathematics. Dirac was so averse to public acclaim that, according to legend, he intended to turn down the 1933 Nobel Prize, relenting only when friends pointed out that he would attract more attention by declining it than by accepting. Nevertheless, a scientist of Dirac's stature could scarcely escape the plaudits of other scientists, which have continued to the present day. In 2002, physicists and mathematicians marked the centennial of Dirac's birth with not one but three memorial conferences.

"Physical laws should have mathematical beauty," Dirac once said—and the equation on the plaque is certainly one of his most beautiful. Dirac's equation is celebrating an anniversary, too: This January marked the 75th year since its first publication (in the *Proceedings of the Royal Society of London*). The equation describes the state, ψ, of a single electron, or (with one additional term) of an electron interacting with a photon of light. Over the last three-quarters of a century, it has inspired major and unexpected developments in both mathematics and physics, such as the discovery of antimatter.

Dirac's equation has to be judged one of the most successful mathematical models ever of a physical phenomenon. And yet it is not even right! The discrepancy between Dirac's equation and the way electrons really behave could provide the first experimental evidence of cracks in the current Standard Model of subatomic particles.

A Splashy Debut

The story begins in 1925, the year that quantum mechanics sprang into existence. Werner Heisenberg in Germany and Erwin Schrödinger in Austria independently proposed novel—and seemingly very different—schemes in which particles (such as photons) and waves (such as light waves) were really the same thing. Heisenberg's paper made its way to Cambridge, where a relativity theorist named R.H. Fowler gave it to his graduate student to read. "What do you think of this? I shall be glad to hear," he wrote across the top.

When the student—Paul Dirac—spoke, the whole world of physics would hear him. Within a few weeks, he had developed a new theory, called transformation theory, that united the apparently irreconcilable ideas of Heisenberg and Schrödinger. After this splashy debut, Dirac rapidly forged ahead into unexplored territory, publishing 11 papers before receiving his doctorate. Because quantum mechanics was utterly new, nearly every paper was a watershed. "[Heisenberg] started the golden age in theoretical physics, and for a few years after that it was easy for any second rate student to do first rate work," Dirac said later with characteristic humility.

No paper had more lasting impact than one he wrote at the end of 1927, called "The Relativistic Theory of the Electron." In this paper, Dirac took a theoretical problem that physicists considered solved and proved that the solvers had missed the point entirely. Unwittingly, he also became the first person to deduce the existence of a new subatomic particle from purely mathematical arguments.

Dirac's Equation and Spin

The theoretical problem Dirac tackled was to unify quantum mechanics and special relativity. Dirac wanted to write the equation of motion of a quantum particle in a way that would respect Lorentz transformations, the fundamental symmetry operation of relativity. This had already been done by Oskar Klein, who (in work published in 1926) derived what is now called the Klein–Gordon equation:

$$(-\hbar^2\partial^2/\partial t^2 + c^2\hbar^2\nabla^2)\psi = m^2c^4\,\psi.$$

(Here c, as usual, is the speed of light, and $\hbar$ is Planck's constant divided by 2π.) Like the classical wave equation, this relates the time derivatives of the particle's motion (the first term) to its spatial derivatives (the second term). Dirac was unhappy with this version, though, because it involved *second* derivatives of the particle's state, ψ. For his transformation theory, he believed that he needed an equation with first derivatives. Thus, Dirac somehow had to take the square root of the operator $(-\partial^2/\partial t^2 + c^2\nabla^2)$ that appears in the Klein–Gordon equation.

Taking the square root of this operator is impossible, in the same sense that taking the square root of -1 is impossible. That didn't stop Dirac. In notation less cryptic than that on his memorial plaque, Dirac came up with the following equation:

$$(i\hbar\partial/\partial t - c\hbar\alpha\cdot\nabla)\psi = \beta mc^2\psi.$$

For the square-rooting to work, α has to consist of three 4 × 4 matrices; β must also be a 4 × 4 matrix, and their products need to satisfy certain simple identities. Most importantly, the unknown quantity ψ in the above equation behaves like the *square root* of a vector. (If you rotate the room you're in by 360 degrees, you don't see any change, and an ordinary vector wouldn't either. But the electron does. Its wave function rotates only half as far, and changes polarity from ψ to $-\psi$. As physicists would say, it goes from "spin up" to "spin down.")

Just as mathematicians invented the imaginary number i to serve as the square root of −1, Dirac had invented a new kind of quantity: the square root of a vector. (He was not quite the first: French mathematician Emil Cartan had come up with the same idea in 1913.) "That a quantity with four components is not a four-vector has never yet happened in relativity," John von Neumann wrote in 1928. The new quantities were soon named *spinors*, a name directly inspired by their physical manifestation.

Why Spin ≠ Rotation

Seventy-five years after Dirac's breakthrough, nearly every popular account of electron spin still describes electrons as if they were rotating billiard balls. And they are all equally wrong. Here are five reasons that this "mental picture," as Dirac would call it, does not conform to reality:

- Electron spin is quantized; the angular momentum of a classical billiard ball is not. Nothing can gradually "slow down" or "speed up" an electron's spin.
- The electron's spin "axis" is completely reassigned by any attempt to measure it. That is, a spin ½ electron will, if measured, also have spin ½ or –½ around the *x*-axis, spin ½ or –½ around the *y*-axis, and spin ½ or –½ around the *z*-axis. (These measurements cannot be performed simultaneously.) By contrast, a billiard ball's axis of rotation is independent of (and may be oblique to) any axis chosen by an experimenter.
- The electron's magnetic moment is two times too large for a spinning ball of charge. (Or its spin is two times too small for its magnetic moment.)
- If an electron were a spinning ball, the linear velocity of its surface would exceed the speed of light.
- Quantum physicists know that an electron does not orbit a nucleus in the same way a planet orbits the Sun. So why should the electron rotate like a planet?

From Spin to Antimatter

An electron is the world's smallest magnet; so much was already known before 1928. The electron's magnetic field causes the lines in the emission spectrum of a hydrogen atom to split—producing the so-called "fine structure" of the spectrum. This fact was explained in 1925 by a classical, non-quantum (or, more accurately, "old" quantum theory) argument: The electron was seen as a spinning ball of charge, and moving charges are known to generate magnetic fields. Unfortunately, this hypothesis didn't square with experimental evidence gathered at the time. Either the electron's magnetic moment was two times too large, or the electron's angular momentum was two times too small.

Dirac's equation destroyed once and for all the idea of the electron as a rotating ball of charge. The equation suggested instead that the electron has a property called *spin*, which is different from physical rotation but manifests itself to us in a similar way, as a magnetic field. Dirac's spinors provided a perfect explanation for the mysterious factor of two. Because physical rotations affect spinors only "half as much" as they affect ordinary vectors, the electron's spin is only half as great as expected. And we should be glad of the difference. All the particles that make up ordinary matter—electrons, protons, and neutrons—have spin ½. A universe without spinors would be a universe devoid of matter.

Because Dirac's spinors have four components, to match the 4 × 4 matrices α and β, it is tempting to describe the electron as a body rotating in four-dimensional space. But that would be misleading. We can never really be sure whether electrons inhabit a four-dimensional world or simply behave as if they do, toting around a four-dimensional representation of themselves as undergraduates used to tote slide rules, prepared (when quizzed by a physicist) to produce the right answers. Dirac himself clearly preferred the latter view. "We cannot form a mental picture [of nature's laws] without introducing irrelevancies," he wrote in 1930. "Any such attempt would be quite opposed to the principles by which modern physics advances. What quantum mechanics does is to try to formulate the underlying laws in such a way that one can determine from them without ambiguity what will happen under any given experimental conditions."

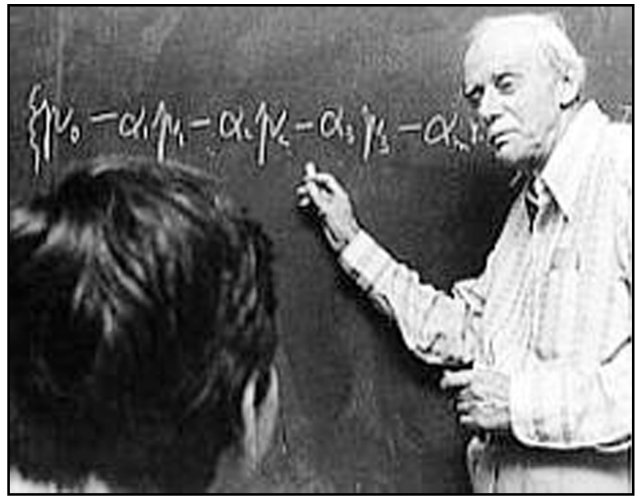
But those four coordinates did have a concrete experimental consequence, which Dirac struggled with at first. Two of the coordinates referred to spin-up and spin-down states of the electron; but what did the other two refer to? Mathematically, they corresponded to *negative-energy* states, which seemed to violate the laws of physics. Dirac proposed that we live in an infinite sea of negative-energy electrons; if one of these electrons were removed, the void left behind would correspond to a positive energy that could be observed in the laboratory. The experimenter would see this hole, which (since a negative electron has been removed) would have a *positive* charge.

Dirac made this proposal—a particle with the mass of an electron but a positive charge—with extreme reluctance in 1931. In those days, quite unlike the present, the proposal of a new particle was seen as a failure to square an experimental observation with the accepted laws of physics. Although Dirac and Heisenberg were good friends, Heisen-berg once wrote to Wolfgang Pauli: "The saddest chapter of modern physics is and remains the Dirac theory." But within a year, Carl Anderson of Caltech had experimentally observed the "anti-electron," or positron as physicists now call it. This was only the tip of an iceberg. Physicists subsequently showed that every elementary particle has an anti-particle, and in 2002 a group at CERN even created measurable amounts of anti-hydrogen—roughly 50,000 atoms. Dirac's discovery was the beginning of a mystery that physicists still debate: Why is there more matter than antimatter in the universe?

Dirac's Legacy

Dirac's introduction of spinors was comparable to the discovery of imaginary numbers in another way, too: Both had consequences far beyond what could be expected from their mathematical definitions.

The ramifications of Dirac's equation continue to be felt to this day, when the greatest interest focuses on what he left out. From both theory and accelerator experiments, it has been apparent since the late 1940s that the "gyromagnetic ratio" between the magnetic moment of the electron and its spin is not exactly 2, as was stated above. According to the latest measurements, the ratio is about 2.002319304376, with some uncertainty in the last digit. Dirac himself actually pioneered the explanation of this deviation, once he had accepted the existence of the positron.



In 2002, physicists and mathematicians marked the centennial of Paul Dirac's birth with three memorial conferences. January 2003 was another Dirac anniversary: His celebrated equation describing the state of an electron had first been published 75 years earlier, in the Proceedings of the Royal Society of London.

When a photon interacts with an electron, a large number of "quantum fluctuations" occur. For instance, the electron can emit and reabsorb a different ("virtual") photon during the interaction. Or it can emit and reabsorb electron–positron pairs. With the theory of quantum electrodynamics, initiated by Dirac, physicists can track all of these interactions and their strengths.

These alternative scenarios, some with even more steps and more exotic particles, all affect the strength of any electron–photon interaction, because that interaction is a "sum over all possible histories." Theorists have worked out thousands of possible histories and their corresponding effects. Amazingly, the theory and the experimental evidence agree up to and including the 11th digit of that 12-digit number.

But in 2002, physicists at Brookhaven National Laboratory reported similar measurements for the muon (a much heavier cousin of the electron). The experiment showed there was trouble with that 12th digit: It lay outside the range predicted by theory. Even though the mismatch is less than one part per billion, it is too big to be explained by chance.

Out of such discrepancies are Nobel prizes made . . . maybe. "This is huge," says William Marciano of Brookhaven. "It's twice as big as the entire contribution of electro-weak physics." In the last year, by Marciano's count, 300 papers have attemped to explain the difference. Many theorists believe it could be the first sign that experimental physics might be ready to move beyond the Standard Model, which has reigned since the 1970s. (The Standard Model or electro-weak theory gave a unified explanation of the electromagnetic and weak nuclear forces.) One popular candidate is a theory called supersymmetry that would double the number of elementary particles.

Yet it is still not clear that there is anything to be explained; possibly there was an experimental error, or an error in any of the thousands of quantum calculations. (One such error, a mistaken minus sign, did turn up in the past year.) Or perhaps the muon has some other, still unknown intrinsic difference from the electron. "I think you have to face up to the difference in mass between the muon and the electron, and understand where that difference comes from," says Roman Jackiw of the Massachusetts Institute of Technology. "I would be disappointed if all you have to do is add more interactions," as in the supersymmetric theory, he says.

Seventy-five years after Dirac wrote down his equation, its consequences are with us every day. Positrons are used in PET scanners, creating images of the human brain. Since 1998, the most advanced disk drives have used electron spin effects to store data. Entire computers using "spintronics" may be just over the horizon. It is a spectacular legacy for a man of legendary modesty. Perhaps Dirac was right: Mathematical beauty is more than its own reward.

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