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Eigenvalues, Anyone?

By Beresford Parlett

IWASEP IV participants gathered in Split, Croatia, in June.

The fourth International Workshop on Accurate Solution to Eigenvalue Problems, briefly IWASEP IV, convened in Split, Croatia, in June, the week after Householder XV (see article in this issue).

The curious use of the word "accurate" in the title is explained by a statement of the theme that launched the series: the description of matrices that define their eigenvalues to high relative accuracy (i.e., eigenvalues near 0 should be determined to the same number of digits as the large ones) and the discovery of algorithms that will deliver such accuracy. This extra demand, going beyond standard "backward stability," is good news for Jacobi methods, which enjoy an advantage over QR in this respect. More recent are the GJG^* decomposition of Veselic and his co-workers and the method of multiple representations developed by Dhillon and Parlett. A concomitant focus is multiplicative perturbation theory, $A \to EAF$ with E and F close to the identity I, which permits relative error bounds to arise in a natural way.

Significant progress has been made on the main goals, and the scope of the workshop has broadened to embrace quadratic eigenvalue problems as well as new iterative methods. The previous workshop, held in Hagen, Germany, in 2000, was enriched by seven participants from the former USSR, and they were missed this time. In fact, most of the 40 participants this year were European; the few Americans were all connected to the letter B (Barlow, Beattie, or Berkeley).

The star attraction of Split (Spalato in Italian) is the huge palace of the Roman emperor Diocletian, an enthusiastic persecutor of Christians. The highlight of our entertainment was a tasty buffet on the evening of Sunday, June 23, served in the recently opened, and deliciously cool, dining room of Diocletian himself. This occurred thanks to our energetic and well-connected local organizer Ivan Slapničar, who did a superb job in looking after us before and during the meeting. The other two members of the organizing committee were Jesse Barlow and Krešimir Veselić.

A brief description of the invited talks gives the flavor of the workshop (while doing a disfavor to the younger participants). In the opening talk Jim Demmel described recent results that fill in squares of his "relative complexity table." The rows designate matrix types, such as Vandermonde, and the columns indicate standard computations, such as inversion or singular values; the squares themselves show whether a guaranteed polynomial-time algorithm exists, is impossible, or is unknown. The catch here is that the output must have some degree of relative accuracy. Actually, there is a different table for each model of computer arithmetic adopted. Many squares of this table have been filled in by core members of these workshops.

New results for generalized Vander-monde matrices made up part of Plamen Koev's dissertation, and he discussed them in one of the later talks. Later in the morning Inderjit Dhillon presented some spectacular results on a 13K $\times$ 13K symmetric tridiagonal matrix and gave a lucid description of the rather complicated method of multiple representations. This work was just beginning at the first of these workshops (in 1996). In the afternoon Zlatko Drmač presented an elegant algorithm for the spectral factorization of a symmetric semidefinite matrix A, but with a special twist: The rank is known either because $A = FF^*$ for a given full column rank F or because a basis for the nullspace of A is given explicitly, as in divergence-free approximations used in electromagnetic studies.

On the second day, Volker Mehrmann discussed quadratic eigenvalue problems whose coefficient matrices are alternately skew Hamiltonian (odd terms) or Hamiltonian (even terms). He described an implicitly restarted skew Hamiltonian Arnoldi-like method that preserves the inherent structure, including a special way to extract invariant subspaces from the Krylov-like subspaces.

Next came Axel Ruhe with a clever algorithm for more general nonlinear eigenvalue problems $A(p)x = 0$. The nonlinearity, in p, is dealt with by a sequence of linearizations using Lagrange interpolation. Each linear problem is solved by Ruhe's rational Krylov subspace algorithm. After all, if systems of equations can be solved directly—not all problems are gigantic—then one can use poles as well as shifts, and the vectors in the so-called "Krylov" subspace are no longer of the form $\phi(A)b$ for polynomial ϕ but are $r(A)b$ where r is rational. The choice of poles is still as much art as science. The afternoon was taken up with an excursion to the island of Brač for swimming and sunning.

Wednesday began with Ilse Ipsen taking the optimal bounds developed by N.J. Lehmann in the 1950s for Hermitian matrices and extending the technique to general square matrices. Now we have disks in the complex plane instead of real intervals, and the radii depend on singular values of certain submatrices (as expected from Lehmann) as well as the departure from normality. These bounds retain optimality for normal matrices. Ipsen also obtained analogous relative bounds in which the singular values are replaced by generalized singular values.

In the afternoon Christopher Beattie gave a theoretical talk that is relevant to the business of model reduction. Often one would like a low-rank approximation to the solution X of a Riccati equation $XHX + AX + XB = G$ when G itself has low rank. The possibility of such an approximation depends on the decay rate of the singular values of X, as the index increases, and we were treated, as expected, to a lucid discussion of this decay rate.

In the final invited talk, on Thursday morning, Hubert Schwetlick turned to the eigenvalue problem for general A of large order. Let u, $||u|| = 1$, be an approximate right eigenvector for a simple eigenvalue λ, and let $\rho = u^*Au$ be its Rayleigh quotient and $r = (A - \rho * I)u$ its residual vector. Schwetlick showed us that the celebrated Jacobi–Davidson correction equation $(I - uu^*)(A - \rho * I)(I - uu^*)s = -r$ for s, to augment the current subspace, is equi-valent to one Newton step from (u,ρ) to $(u + s, \rho + \alpha)$, by solving $(A - \rho * I)s - u\alpha = -r$, $u^*s = 0$. He went on to give a beautiful exposition of the efficiency of more general projectors, such as the approximate spectral projector for λ, of the use of Ritz–Petrov values instead of just Ritz values, and of block methods for dealing with clusters of close eigenvalues.

Last but not least, the posters, all eight of them, were on view throughout the workshop and were studied by most participants. The prize for the best poster went to Bor Plestenjak and Michiel Hochstenbach, for "A Jacobi–Davidson Type Method for a Right Definite Two-parameter Eigenvalue Problem." It is worth noting that the two winners were inspired to work on this problem when they met at the Hagen workshop in July 2000.

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