# **Stirs, Stars, Bugs, and Bounces At SIAM Dynamical Systems Conference**

By Barry A. Cipra

Five hundred mathematicians and their collaborators from a variety of disciplines met in Snowbird, Utah, May 20–24, for the Sixth SIAM Conference on Applications of Dynamical Systems. As remnants of winter snow melted from the nearby ski slopes, attendees at the conference heard talks on topics ranging from celestial mechanics to the scurrying of cockroaches.

## Stirred, Not Shaken

Jerry Gollub, a physicist at Haverford College, described recent advances in the study of transient mixing. The basic problem is an old one: What happens when you stir a splash of cream into a cup of coffee? More technically, if you start with a localized impurity in a moving fluid, how is homogeneity eventually reached?

Researchers are getting insights from some novel experiments. In particular, Gollub's group has studied a thin-film fluid in which the flow is driven by an array of magnets and the impurity is a fluorescent dye. The effectively two-dimensional system is governed

by three dimensionless quantities: Reynolds number Re = LV/v (*L* being the spacing of the magnets and *V* the fluid velocity), path length p = V/fL (*f* being the forcing frequency), and Peclet number Pe = LV/D (*D* being the diffusion coefficient).

The experiment starts with the dye isolated in the left half of a square tank. The barrier is removed, and the dye begins to mix by chaotic advection. A video shows an elaborate pattern of striations developing in the moving fluid. Remarkably, the striations don't broaden, even though diffusion is mixing the dye into the rest of the fluid at the smallest scales.

Even more remarkably, a series of snapshots taken at the forcing frequency—in effect, a Poincaré return map—reveal an invariant pattern, which only slowly fades as the fluid becomes homogeneous. "Even in 150 cycles, the process is not yet complete," Gollub notes. If you weren't told that you were watching a laboratory experiment, you'd swear it was a carefully constructed computer simulation. This work was described in *Nature* (Vol. 401, 1999, page 770).

More recently, to analyze what's going on, Gollub's postdoc Greg Voth added approximately 400 fluorescent particles to the fluid and traced their motion through 240 periods of the magnetic forcing, taking 50 pictures per period—12,000 images in all. A color-coded, animated map showing how each particle moves in one forcing cycle calls to mind Van Gogh's *Starry Night*, and reveals the presence of two kinds of fixed points in the fluid. The fluid is strongly stretched near the "hyperbolic" fixed points, while it simply circulates gently near the "elliptic" fixed points (see Figure 1). Additional work, done in collaboration with George Haller of Brown University, has used direct measurements of the stretching in the flow to reveal the stable and unstable manifolds associated with these fixed points. In particular, the unstable manifolds constitute "attracting lines" that organize



**Figure 1.** Poincaré meets Van Gogh. The motion of a periodically forced fluid can be tracked by means of fluorescent particles. This portion of a black and white version of the Poincaré map shows lines connecting the measured position of each particle with its position one period later. A hyperbolic point is clearly visible in the upper part of the plot, and two elliptic and two hyperbolic points can be seen in the lower part.

the evolution of the striations in the fluorescent dye: The contours lie generally parallel to the attracting lines.

In the future, Gollub says, the group plans to look at the possibility of predicting mixing rates—a parameter of considerable importance in applications—by measuring stretching rates along the trajectories. They also hope to study mixing in turbulent flows, where the velocity field is not time-periodic. Enough problems seem to be arising in the field to keep things stirred up for some time to come.

## **New Dance Craze**

Compared with the murky continuum of fluid mechanics, the stately progression of heavenly bodies would seem to offer little challenge and few surprises for the modern applied mathematician. As anyone familiar with the famous three-body problem knows, however, celestial mechanics is far from a closed book. Richard Montgomery of the University of California at Santa Cruz presented

some of the latest discoveries in the repertoire of gravitational attraction. He and Alain Chenciner at the University of Paris VII, along with Carles Simó of the University of Barcelona and other colleagues, have found a slew of new periodic orbits, some with hundreds of bodies chasing each other in a never-ending game of follow-the-leader.

The new closed orbits are essentially the first such solutions to be found in more than a hundred years. In 1878, the mathematical astronomer George Hill showed that "tight binaries," in which two nearby masses circle one another while at the same time circling a third, distant mass (think earth–moon–sun), can be periodic. (The term "circle" is meant loosely; the actual paths are not described by any simple equation.)

A century earlier, Euler and Lagrange had given two other solutions. In Euler's, the three masses are always collinear, while for Lagrange they maintain the shape of an equilateral triangle. When all three masses are equal, Euler's solution has the outer two tracing a circle at opposite ends of a diameter, with the third mass motionless at the center; Lagrange's solution simply spins the equilateral triangle about its center. In general, these 18th-century solutions have each mass moving along an elliptical path, almost as if each were in its own two-body world.

The first new orbit—and the only new one, so far, for which a rigorous proof has been given—has three equal masses tracing out a figure eight (Figure 2). It y



Figure 2. Periodic orbit. Three equal masses loop endlessly in a figure eight. At six equally spaced times during the orbit, the masses are collinear. An animated version, created by graduate student Mike Wessler at MIT's Artificial Intelligence Laboratory, is available at http://www.ai.mit.edu/ people/wessler/halo/rmont.html.

has been given—has three equal masses tracing out a figure eight (Figure 2). It was found numerically in 1993, by Cris Moore at the Santa Fe Institute. Montgomery and Chenciner rediscovered it two years ago and have proved its existence and stability.

The orbit's stability was "a big surprise," Montgomery says. The precise result is that the orbit is KAM-stable (the acronym refers to the Kolmogorov–Arnold–Moser theorem). KAM orbits are not stable in the usual sense of dynamical systems. Rather, they are stable in a probabilistic sense: The smaller the perturbation from the initial condition, the higher the probability that the trajectory will never escape from the orbit. Moreover, the trajectories that do escape take an exponentially long time to do so—so long that it is very difficult, if not impossible, to tell if a given small perturbation is headed that way.

Montgomery and Chenciner's existence proof has three key ingredients. The first is the use of the principle of least action to formulate the problem in terms of the calculus of variations. The second is a reconception of the orbit in terms of a path in "shape space." For three bodies, the shape space turns out to be the sphere: At each moment, the three bodies determine a triangle whose shape is specified by two parameters, which can be turned into a latitude and a longitude. Although the obvious choice of parameters might seem to be two of the angles, the actual identification is done differently (Figure 3). In particular, latitude (or, more precisely, its cosine) is determined by the ratio of the triangle's area to the area of an equilateral triangle with the same perimeter. (Which hemisphere, north or south, depends on the orientation, clockwise or counterclockwise, of the labelling of the vertices.) Thus, the

north and south poles of the sphere correspond to equilateral triangles, while the equator corresponds to triangles with zero area, i.e., collinear triangles. Six meridians, spaced 60 degrees apart, correspond to isosceles triangles. Any trajectory of the three bodies can be projected down to a path in the shape space. Conversely, any path in the shape space can be lifted in various ways to a trajectory. The trick is to see whether any of them obey Newton's law.

The figure-eight orbit has an abundance of the existence proof's third ingredient: symmetry. At six equally spaced points in time during the orbit, the three bodies are collinear, with one of the three at the figure eight's intersection point and the other two on opposite lobes. And midway between colline-ations, the three bodies take the shape of an isosceles triangle. This means that the entire orbit can be derived from just the first twelfth of it. This was crucial, Montgomery says, in the final step of the proof: showing that the three bodies don't collide in the course of their newfound orbit.

When Montgomery first found the new orbit, he did not realize that all three masses follow the same trajectory. It was Chenciner who discovered this surprising property. Simó, in careful numerical analysis, corroborated it. The researchers have dubbed a solution of this type a "choreography." Twyla Tharp, move over.

The simulations also provided the first hint of the orbit's stability. The computations were "crucial in making us trust the [theoretical] results," Montgomery recalls. But now, almost two years after the initial discovery, the numerics have taken the lead. For the *N*-body community, there are millions of new periodic orbits to sort through.



**Figure 3.** Shape space. Points on a sphere correspond to oriented triangles, with equilateral triangles at the poles and collinear triangles along the equator. With luck and analysis, a closed curve in shape space can be associated with a periodic orbit of three bodies.

It was as if a dam had burst, Montgomery says. The first advance was a four-body choreography discovered by Joseph Gerver of Rutgers University, in Camden, New Jersey. Gerver's orbit belongs to a shape space of parallelograms; the four masses trace out a double figure eight (see Figure 4). Spurred by Gerver's result, Simó "went crazy," Montgomery jokes. "He stayed up till 5 in the morning for about a month straight," by the end of which he had numerical *N*-body choreographies with *N* up to 799. More

recently, he's gone back to the N = 3 case and found nearly a million new solutions. "He's far outstripped our ability to prove things right now," Montgomery admits.

The new orbits raise a host of new mathematical questions. They also pose at least one for astronomers: Do any of these cosmic choreographies actually occur? The figure eight's stability suggests that it might. However, its domain of stability—mainly the extent to which the three masses can be unequal—is very small. According to Montgomery, numerical experiments done by Douglas Heggie of the University of Edinburgh suggest that the probability of such a *pas de trois* "is somewhere between one per



**Figure 4.** Periodic parallelogram. Four equal masses loop endlessly in a double figure eight, discovered by Joseph Gerver. At each moment, the four bodies are at the vertices of a parallelogram.

galaxy and one per universe." In other words, one way for extraterrestrials to advertise their existence (and demonstrate their stellar engineering skills) would be to choreograph a couple of constellations. As they say, the truth is out there.

#### **Keeping Pace**

Berkeley biologist Bob Full has something in common with the proverbial topologist who can't tell the difference between a donut and a coffee cup. In Full's case, the confusion concerns a person and a cockroach.

What Full has in mind is not a Kafkaesque comparison of appearance or human value, but merely—and surprisingly—that when it comes to ambulation, *Homo sapiens* and *Periplaneta americana* are, biomechanically speaking, a lot alike. Indeed, virtually all the pedestrian critters studied by Full and colleagues, from kangaroos to millipedes, adhere to a unique biomechanical strategy for getting around: In effect, their legs act as pogo sticks.

Full's group at Berkeley, dubbed Poly-PEDAL (for Performance, Energetics, Dynamics of Animal Locomotion), has studied dozens of insects as they run on tiny treadmills,

over rugged obstacle courses, and across force fields of unflavored gelatin. (Under polarized light, the gelatin records the forces exerted by individual footsteps.) In collaboration with robotics expert Dan Koditschek of the University of Michigan and mathematicians John Guckenheimer of Cornell and Philip Holmes of Princeton, Full has developed new mathematical models that describe the stability and control of animal gaits.

Modeling, Full says, makes it possible to formulate precise hypotheses about the ways animals control their forward motion. One of the major discoveries is that the brain plays a different role in motion than was previously thought. The old theory put the brain (and its subsidiaries) very much in the driver's seat: Forward motion was maintained by constant monitoring of sensory data and correction of deviations through an elaborate feedback system. The new theory relies instead on feedforward control. "Basically, the control mechanism can be embedded in the animal itself," Full explains. In effect, the brain simply says "keep going," and nature's fine-tuning of muscle and bone does the rest.

Full and colleagues call this kind of mechanical feedback "preflexes," because it operates automatically, even before the nervous system's reflexes kick in. One of the most convincing demonstrations (and a crowd favorite whenever the video clip is played) has a cockroach running on a treadmill with a tiny "cannon" glued to its back, pointing to the side. When the cannon fires (it's detonated electrically, like the bridges blown up in World War II movies), the roach is jolted in equal and opposite reaction, but the hardy product of evolution scarcely falters: Within a couple of steps, it's back in full stride.

The new discoveries about animal gaits are motivating new approaches in robotics, Full says. Koditschek's group, for example, has incorporated feedforward techniques in a six-legged robot called RHex. With carefully calibrated springiness built into its legs and control algorithms that take advantage of the mechanics, RHex is capable of thrashing its way over various kinds of terrain, including dense brush and rocky slopes. (The latter, Full says, really caught the eye of NASA engineers, who had dismissed the idea of sending a legged robot to Mars.) Video clips are available at the RHex Web site, http://ai.eecs.umich.edu/RHex/. With progress being made at a rapid pace, Full sees a promising future for the advancing science of neuromechanics.

### Follow the Bouncing Ball

Thomas Vincent, a mechanical engineer at the University of Arizona, studies a control problem that might seem simpler than six-legged motion: the bouncing of a ball on a vibrating plate. But looks can be deceiving. Even when the ball is impaled on a rod, there's a lot of chaos to be kept in check. Vincent demonstrated the dynamics at the conference poster session and described the control algorithms during a minisymposium.

Vincent's poster, done in collaboration with Brad Paden of the University of California, Santa Barbara, was one of four winners of the James Yorke Red Sock award. (Yorke is director of the Institute for Physical Sciences and Technology at the University of

Maryland, and a prominent figure in dynamical systems. He is also known for wearing bright red socks. The color, he explains, facilitates sorting of the family laundry.) The other winners were John Harlim and William Langford (University of Guelph), for "The Codimension Three Cusp Hopf Birfurcation"; Martin Homer (University of Bristol), for "Nonlinearity and Asymmetry in the Vibrations of the Inner Ear"; and Tyler McMillen and Alain Goriely (University of Arizona), for "Tendril Perversion in Intrinsically Curved Rods."

The basic dynamics of the ball-and-plate system are fairly straightforward: What goes up must come down, and then it goes back up again (Figure 5). The theory was exposited by Guckenheimer and Holmes in their classic 1983 text,



**Figure 5.** Follow the bouncing ball. The dynamics of a ball (top, piecewise-parabolic curve) bouncing off a vibrating plate (bottom, sinusoidal curve) are highly complex. At a frequency of 30 radians/second, any perturbation of an unstable period-1 solution leads to a stable period-2 solution.

Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. The pertinent variables are the amplitude A and frequency  $\omega$  of the vibrating plate and its phase angle  $\phi_i$  at the time of the *j*th bounce. The coefficient of restitution e (the bounciness of the ball) is also an important parameter. If the height of the bounce is large compared with the amplitude of the plate's vibration, the time between bounces is approximately  $2V_i/g$ , where  $V_i$  is the departure velocity at the time of the *j*th bounce. This updates the phase angle to  $\phi_{i+1} = \phi_i + 2\omega V_i/g$ . The new velocity,  $V_{i+1}$ , can also be worked out as  $V_{j+1} = eV_j +$  $A(1 + e)\omega \cos(\phi_i + 2\omega V_i/g)$ . (The plate is assumed to be "infinitely" heavy compared with the ball.)

In a paper published last year in the *International Journal of Bifurcation and Chaos*, Vincent and Alistair Mees of the University of Western Australia analyzed what happens when the frequency is varied with each bounce, so that  $\omega$  is replaced by  $\omega_j$ . They also took into account the ratio *r* of the ball-to-plate masses, which changes the terms *e* and 1 + e in the velocity formula to



**Figure 6.** Getting control. At 30 radians/second, a data-based approximation and either LQR control (top) or greedy control (bottom) can be used to stabilize the ordinarily unstable period-1 solution.

(e - r)/(1 + r) and (1 + e)/(1 + r), respectively. Letting  $\omega_0$  denote the nominal frequency of the system and defining  $\psi_j$  as  $2\omega_0 V_j/g$ , they obtain a system of the form  $\phi_{j+1} = \phi_j + (\omega_j/\omega_0)\psi_j \psi_{j+1} = a\psi_j + b\omega_0\omega_j \cos\phi_{j+1}$ , where *a* and *b* are considered fixed parameters (that is, not dependent on frequency). It's relatively straightforward to solve for periodic solutions, around which the map can be linearized.

In Vincent and Mees's analysis, *A* is fixed at 1.3 cm, *e* at 0.8, and *r* at 1/26. This corresponds to a = 0.73333 and b = 0.004594 sec<sup>2</sup>. They ran simulations with  $\omega_0$  at 22 and 30 rad/sec. In both cases the system, theoretically, has a period-1 solution in the absence of control (i.e., for  $\omega_j = \omega_0$ ). The solution is stable for  $\omega_0 = 22$ , and unstable for  $\omega_0 = 30$ . But by adding control—that is, by varying  $\omega_i$  appropriately—they were able to create stability in the latter case and increase it in the former.

They tested two control algorithms: an LQR (linear-quadratic-regulator) method and a "greedy" method that seeks to minimize the mean square error just one step ahead (instead of looking further into the future). The greedy algorithm works particularly well in the  $\omega = 22$  case, more than doubling the range of initial heights for which the period-1 solution is asymptotically stable. And both algorithms turn the unstable,  $\omega = 30$  solution into a nicely stable, periodic pattern. However, to make things work, especially in the higher-frequency case, they found it necessary to replace the linear map derived from the high-bounce approximation with a data-based linear approximation (Figure 6). (The heights of the bounce corresponding to the fixed points for  $\omega = 22$  and 30 are approximately 11 and 6.5 cm, respectively. Neither value is large enough compared with the amplitude A = 1.3 cm to completely justify the high-bounce approximation.)

More recently, the researchers have run experiments with an actual bouncing ball apparatus, built by Paden's company, Magnetic Moments. The laboratory results, Vincent reports, show that the control algorithms work as well in practice as they do in simulations. Visitors to the poster at Snowbird were able to see for themselves: A steel ball bounced hypnotically up and down. For those who didn't make it to the meeting, there's a video clip at Vincent's Web site (http://www.ame.arizona.edu/faculty/vincent/tincent.html).

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