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# Control of Mechanical Systems Subject to Unilateral Constraints

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Mechanical systems subject to non-smooth impacts from unilateral constraints are interesting not only because of their many technological applications—which include walking, hopping, and juggling robots, gear- and cam-based mechanisms, hammering tasks, and manipulation problems related to spacecraft—but also because of the challenging new control problems they suggest. Many available control algorithms, for example, like those based on classical Lyapunov stability theory, require uniqueness and continuity with respect to the initial conditions of the solution to the dynamical system being controlled; these properties can be lost in the presence of unilateral constraints, even in the simple case of linear equations and constraints.

Such systems have appeared in the scientific literature since the early works of Newton and Hertz. For an extensive review of existing results, the interested reader is referred to [2].

In the last few years, much research has been devoted to the control of so-called hybrid dynamical systems [1, 4, 5]—systems whose description includes both a "continuous-time" and a "discrete-time" component. Mechanical systems subject to non-smooth impacts represent a subclass of hybrid systems: Continuous-time descriptions can be given between any two adjacent impact times, but the overall behavior of the system can be captured only if some properties of the phenomena are taken into account. In this way, the postimpact state of the system can be computed as a function of the preimpact state. A number of problems concerned with the modeling of impact events are still unsolved [2].

The Hamiltonian structure that can be attributed to unconstrained mechanical systems, combined with the special type of discontinuities generated by the impacts, makes it easier to deal with such a subclass than with the whole class of hybrid dynamical systems. The results obtained are thus more powerful. In this article, we discuss control problems that cannot be solved within the framework of classical control theory, requiring special formulation and/or ad hoc mathematical tools.

## Mechanical Systems with Inequality Constraints

Throughout the article, we consider finite-dimensional mechanical systems that can be described by a vector $\mathbf{q}(t) \in \mathrm{R}^n$ of generalized coordinates. If $\mathbf{q}(t)$ is constrained to belong to an admissible region, say

$$\mathcal{A} := \{\mathbf{q} \in \mathrm{R}^n : f_i(\mathbf{q}) \le 0, \quad i = 1, 2, \ldots, m\}, \tag{1}$$

there can be times $t_c$ at which $\mathbf{q}(t)$ is not differentiable, i.e., impact times. If $f_i(\mathbf{q}(t)) = 0$ for some $i \in \{1, 2, \ldots, m\}$ and for some $t \in \mathrm{R}$, then some parts of the mechanical system are, at such times, in contact with themselves or with the external environment. An impact can occur at a certain time $t_c \in \mathrm{R}$ only if, at such a time, $f_i(\mathbf{q}(t_c)) = 0$ for some $i \in \{1, 2, \ldots, m\}$. If $f_i(\mathbf{q}(t)) = 0$ for more than one index $i \in \{1, 2, \ldots, m\}$ and for the same time $t$, then there are multiple contacts at time $t$.

By assuming that the impacts do not cause instantaneous loss of energy, the method of *Valentine variables* makes it possible to model mechanical systems subject to inequality constraints [3] by means of the Hamilton principle. Such a method consists of a double transformation of the inequality constraints, first into equality constraints and then, for convenience, into differential constraints.

If $T(\mathbf{q}(t),\dot{\mathbf{q}}(t)) = (1/2)\, \dot{\mathbf{q}}^T(t)\, \mathbf{B}(\mathbf{q}(t))\, \dot{\mathbf{q}}(t)$ is the kinetic energy and $U_{tot}(\mathbf{q}(t)) = U(\mathbf{q}(t)) - \mathbf{q}^T(t)\; \mathbf{E}\, \mathbf{u}(t)$ the total potential energy of the mechanical system (taking into account the action of the vector $\mathbf{u}(t) \in \mathrm{R}^p$ of the generalized control forces), we can denote the Lagrangian function as $L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) := T(\mathbf{q}(t),\dot{\mathbf{q}}(t)) - U(\mathbf{q}(t))$. The differential equation describing the motion of the system is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} + \sum_{i=1}^{m} \dot{\lambda}_i \mathbf{J}_i(\mathbf{q}) = \mathbf{E}\mathbf{u}, \tag{2}$$

where $\dot{\lambda}_i$ are the derivatives (defined only in the distributional sense) of the Lagrange multipliers used to account for the differential constraints and $\mathbf{J}_i(\mathbf{q})$ denotes the (column) Jacobian vector of $f_i(\mathbf{q})$. For computation of the system behavior from given initial conditions, equation (2) is integrated in the intervals between impact times, with the following *Erdmann–Weierstrass corner conditions* satisfied at the impact times:

$$\frac{1}{2}\dot{\mathbf{q}}^T\left(t_c^-\right)\mathbf{B}\left(\mathbf{q}\left(t_c\right)\right)\dot{\mathbf{q}}\left(t_c^-\right) = \frac{1}{2}\dot{\mathbf{q}}^T\left(t_c^+\right)\mathbf{B}\left(\mathbf{q}\left(t_c\right)\right)\dot{\mathbf{q}}\left(t_c^+\right), \tag{3a}$$

$$\mathbf{B}\left(\mathbf{q}\left(t_c\right)\right)\dot{\mathbf{q}}\left(t_c^-\right)+\sum_{i=1}^{m}\lambda_i\left(t_c^-\right)\mathbf{J}_i\left(\mathbf{q}\left(t_c\right)\right) = \mathbf{B}\left(\mathbf{q}\left(t_c\right)\right)\dot{\mathbf{q}}\left(t_c^+\right)+\sum_{i=1}^{m}\lambda_i\left(t_c^+\right)\mathbf{J}_i\left(\mathbf{q}\left(t_c\right)\right). \tag{3b}$$

The Erdmann–Weierstrass corner condition (3a) states that the kinetic energies immediately before and after an impact must be equal, whereas equation (3b) relates the jump $\dot{\mathbf{q}}(t_c^+) - \dot{\mathbf{q}}(t_c^-)$ of the generalized velocities to the jump $\lambda_i(t_c^+) - \lambda_i(t_c^-)$ of the Lagrange multi-pliers. In the case of single impacts, we can obtain unique solutions to equations (3a)–(3b) by requiring that the mechanical system stay within the admissible region; hence, such equations allow the postimpact velocities to be determined as functions of the preimpact velocities and of the system configuration.

Some control problems can be solved for mechanical systems subject to unilateral constraints without accounting for the constraints in the design of the control law. For one such problem—the stabilization of an admissible equilibrium configuration—we assume that all the state variables $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ are mea-sured and apply a standard "proportional-derivative" control law.

Under mild assumptions, it can be proved [3] that, for practical purposes, the equilibrium point of the "closed-loop system" is asymptotically stable. Indeed, such an equilibrium point has a sort of stability property that is limited to the generalized coordinates $\mathbf{q}$, the generalized velocities $\dot{\mathbf{q}}$, the classical attractivity property with respect to $\mathbf{q}$, $\dot{\mathbf{q}}$ and the generalized reaction forces $\dot{\lambda}_i$ (which are necessarily excluded from the stability requirement). Such properties have been demonstrated by using the total energy of the system (including a term due to the proportional action in the control law) as a Lyapunov functional and making use of a suitable extension of LaSalle's theorem.

In control theory, LaSalle's theorem represents a powerful result from Lyapunov stability theory, allowing the guarantee of asymptotic stability when it is possible to prove only that the time derivative of a candidate Lyapunov functional is negative semidefinite, and not negative definite. An extension proved in [3] provided a similar result for systems in which some of the state variables (the velocity variables) are subject to jumps because of the impulsive values of some of the other state variables (the generalized contact forces).

## Velocity Observers for Systems Subject to Non-smooth Impacts

In the previous section, we described a problem involving non-smooth impacts that can be solved with the same control law used to solve the corresponding problem for an unconstrained system. This is not the case for the problem of estimating velocity variables when they cannot be measured directly.

In general, for a dynamical system whose state $\mathbf{x}(t)$ cannot be measured directly, the problem of asymptotic state estimation consists of designing a dynamical system (called the "state observer") that has as input the measured output of the system (the variable we call $y(t)$) and that produces as output an estimate $\hat{\mathbf{x}}(t)$ of the state $\mathbf{x}(t)$ such that the estimation error $\mathbf{x}(t) - \hat{\mathbf{x}}(t)$ asymptotically tends to zero when the time variable $t$ tends to $+\infty$. A general design procedure whose efficiency is well known, at least for linear time-invariant dynamical systems, consists of designing the state observer as a dynamical system whose dynamical equations are the same as those of the system whose state is to be estimated, with the addition of a suitable correction term, proportional to the difference between the measured output and its estimate. State observers designed according to such a principle are known in the literature as Luenberger observers.

For unconstrained linear mechanical systems, such a procedure can be used to design velocity observers. Observers of this type cannot be used as such in the presence of impacts, however (unless the magnitude of the jumps in the velocities occurring at the impact times tends to zero for large times). Therefore, "ad hoc" observation algorithms must be used in presence of non-smooth impacts. In this section, we use a simple example to discuss this problem.

Consider a dimensionless ball with unitary mass, constrained to move along a vertical line in the upper half-space delimited by a perfectly rigid horizontal plane, and denote by $-q(t)$ its distance from the plane. Assuming that there is no dissipation at the impact times or during the free motion of the ball, such a dynamical system can be written in state–space form (with only the position of the mass to be measured) as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}g, \quad t \in (t_i, t_{i+1}),\ i \in \mathrm{Z}^+ \tag{4a}$$

$$y(t) = \mathbf{C}\,\mathbf{x}(t), \quad t \in \mathrm{R},\ t \geq t_0 \tag{4b}$$

$$\dot{q}(t_i^+) = -\dot{q}(t_i^-), \quad i \in \mathrm{N} \tag{4c}$$

where $t_0$ is the initial time; $t_i$, $i \in \mathrm{N}$ are the impact times; $\mathbf{x}(t) := [q(t)\ \dot{q}(t)]^T$, $g$ is the gravity acceleration; $y(t)$ is the measured output $y(t) = q(t) = x_1(t)$; and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are suitable matrices.

In the absence of impacts, an asymptotic observer for system (4a), (4b) can be given as follows:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\,g + \mathbf{K}\left[y(t) - \mathbf{C}\,\hat{\mathbf{x}}(t)\right], \tag{5}$$

where

$$\hat{\mathbf{x}}(t) := \begin{bmatrix} \hat{q}(t) \\ \hat{v}(t) \end{bmatrix}$$

is the estimate of $\mathbf{x}(t)$ and

$$\mathbf{K} := \begin{bmatrix} k_i \\ k_2 \end{bmatrix}$$

with $k_1$, $k_2$ being positive real numbers.

It is easy to see that this dynamical system cannot be taken as an observer for the mass in the presence of an infinite sequence of impacts (as implied by the gravity acceleration), as at each impact time we have a nonzero jump in $\dot{q}(t)$ that can be reproduced in $\hat{v}(t)$ only asymptotically. In a simulation performed to show the drawbacks of the dynamical system (5) as a velocity observer, the gains are given as $k_1 = 3$ and $k_2 = 2$ and the initial conditions as $q(0) = -2, \dot{q}(0) = 10$, $\hat{q}(0) = -2$, $\hat{v}(0) = 0$. Figure 1 shows the behavior of position $q(t)$ (with the dashed line delimiting the admissible region), velocity $\dot{q}(t)$, and state variables $\hat{q}(t)$, $\hat{v}(t)$ (bold lines) of the dynamical system (5); $q(t)$ and $\hat{v}(t)$ tend to a limit-cycle.

The alternative observer proposed here for system (4) is

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\,g + \mathbf{K}\left[y(t) - \mathbf{C}\,\hat{\mathbf{x}}(t)\right],$$
$$t \in \left(t_{i,}t_{i+1}\right), i \in \mathbb{Z}^+, \tag{6a}$$

$$\hat{v}\left(t_i^+\right) = -\hat{v}\left(t_i^-\right), \quad i \in \mathbb{N}, \tag{6b}$$

a slight modification of (5) that takes into account the impacts through equation (6b).

A simulation was performed with the same gains and initial conditions used for the observer (5). Figure 2 clearly shows the fast convergence of the estimated state ($\hat{q}(t)$ and $\hat{v}(t)$) toward the actual state ($q(t)$ and $\dot{q}(t)$) of the mechanical system, despite the impacts affecting the mass behavior (compare the plots of Figure 1 with those of Figure 2).

The solution proposed here for the simple example of the bouncing ball can be used for any linear mechanical system, provided that all position variables are measured and that the impact times are detected in real time (although the estimates obtained are "robust" with respect to small delays in the detection of impacts).

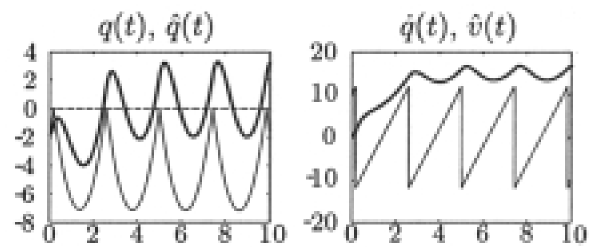


Figure 1. *Drawbacks of a classical velocity observer: plots of $q(t)$, $\hat{q}(t)$ [m] and $\dot{q}(t)$, $\hat{v}(t)$ [m/s] versus t [s]; the estimates are reported in bold.*

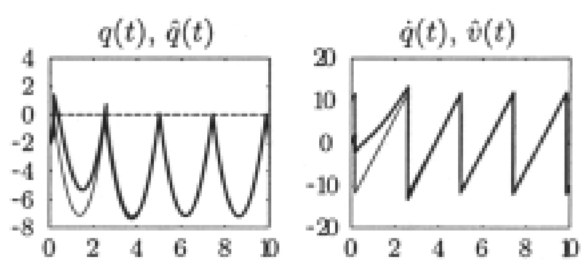


Figure 2. *Behavior of the observer (6): plots of $q(t)$, $\hat{q}(t)$ [m] and $\dot{q}(t)$, $\hat{v}(t)$ [m/s] versus time t [s]; the estimates are reported in bold.*

## Control of Underactuated Systems Through Impacts

In the preceding sections, we described two standard problems for unconstrained systems, showing how they can be solved despite the presence of the inequality constraints and the corresponding impacts. In this concluding section, we present a problem of a completely different type, showing in a simple example how impacts can be used to regulate the position of an underactuated mechanical system. Such a system—that is, a system in which the number of scalar control inputs is smaller than the number of degrees of freedom—would otherwise be uncontrollable.

Consider the mechanism shown in Figure 3, which consists of two infinitely rigid mating gears. Let $x(t)$ and $y(t)$ be the angular positions of the gears. Assume that there is a backlash between the two mating gears, and that the measuring unit used for $x(t)$ and $y(t)$ is such that this backlash is equal to 1. The positions of the two gears are thus constrained by the two inequalities

$$x(t) - y(t) \le 0, \tag{7a}$$
$$y(t) - x(t) - 1 \le 0. \tag{7b}$$

For simplicity, we assume that the rotational inertia of each gear is equal to 1, and that only the second gear is actuated: The first one is almost always in free motion, if we exclude the times at which impacts occur. Denote by $u(t)$ the torque exerted at time $t$ on the rotation axis of the second gear. The Euler–Lagrange equations (to be satisfied between any two consecutive impact times) are

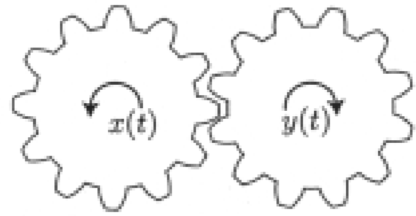


Figure 3. *Two mating gears with backlash.*

$$\ddot{x}(t) + \dot{\lambda}_1(t) - \dot{\lambda}_2(t) = 0,$$

$$\ddot{y}(t) + \dot{\lambda}_2(t) - \dot{\lambda}_1(t) = u(t),$$

where $\dot{\lambda}_1(t)$ and $\dot{\lambda}_2(t)$ are the reaction torques resulting from possible contacts and impacts resulting from the first and the second constraint, respectively. The postimpact velocities can be expressed as functions of the preimpact velocities, whether the impact is due to the first or the second constraint:

$$\begin{aligned} \dot{x}(t_i^+) &= \dot{y}(t_i^-), \\ \dot{y}(t_i^+) &= \dot{x}(t_i^-). \end{aligned}$$

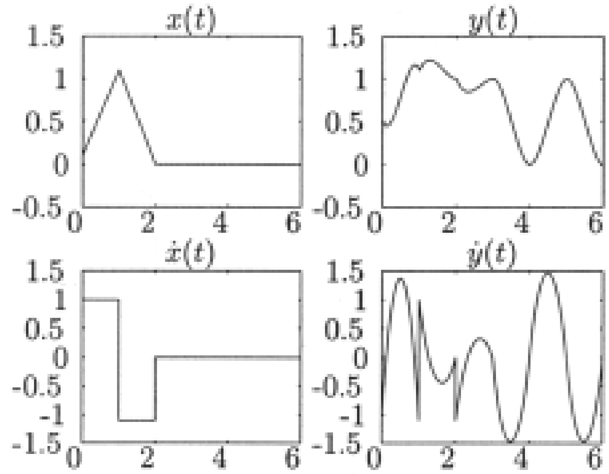


Figure 4. *Simulation results for the regulation of the nonactuated gear.*

The control problem to be solved is the dead-beat regulation to zero of the position of the nonactuated gear. The goal, in other words, is that, independent of the initial conditions, there exists for the closed-loop system a finite time $T_f$ such that $x(t) = 0$ for all $t \geq T_f$; the impacts with the actuated gear are the only way to obtain this result. Starting from the equations above, a feedback control law has been designed, based on a discrete-time interpretation of the dynamical system to be controlled. The results of a simulation of the closed-loop system are reported in Figure 4; from the initial conditions $x(0) = 0.1, y(0) = 0.5, \dot{x}(0) = 1, \dot{y}(0) = -1$, the desired position $x = 0$ is attained in 2 seconds.


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