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A Dynamics Group Looks at the Ocean

By Christopher K.R.T. Jones

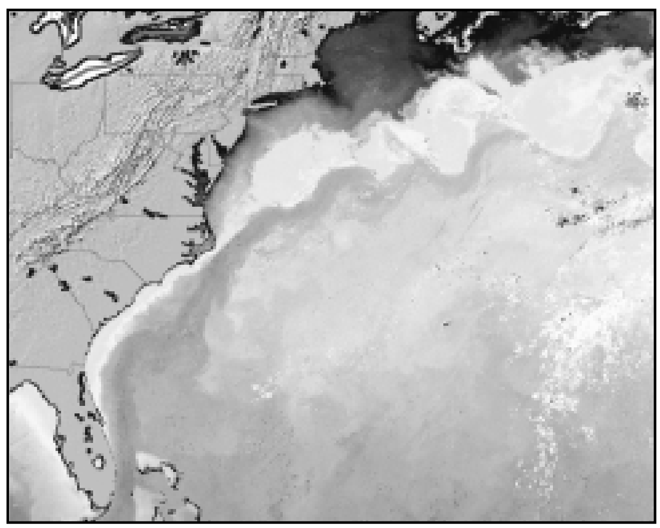
Anyone who has spent time in Northern Europe, with its far warmer climate than justified by its latitude, has experienced the impact of the Gulf Stream. The Gulf Stream carries warm equatorial water into the cold North Atlantic, dramatically altering the ocean, as well as the atmosphere, during its passage to the northeast. This process by which temperature, salinity, and other ocean properties are transported is critical to the overall balance of the ocean.

The significance of the Gulf Stream and its offspring, the North Atlantic Current, for the climate of Northern Europe and the rest of the world is described in an interesting and provocative article by William Calvin [3]. Calvin points particularly to the important consequences of potential changes in the transport of salty, warm water for the climate as we consider the possibilities of global warming or drastic cooling.

The Gulf Stream is only a rather dramatic and persistent example of what occurs in the ocean at all scales. A cartoon we might imagine would show long worms (currents) and rings (eddies) swimming around in the ocean, all acting as carriers of ocean "information" from one place to another. To carry the metaphor one step further, we could ask how, and in what quantities, these coherent "information carriers" impart their properties to the ambient ocean as they pass through.

Backdrop for a Collaboration

The way oceanographers deal with these issues is very close in spirit to the approach taken by researchers in modern, geometric dynamical systems. The motion of an individual fluid particle can be viewed as the trajectory of a dynamical system, with the differential equation being given by the velocity field of the fluid flow. There is a striking match between the qualitative view of dynamical systems, with its focus on gross properties of the trajectories, and the desire to understand "bulk" transport in the fluid flow. This observation was exploited in mixing studies of fluid flow in which it was shown how chaotic dynamics can explain stirring and mixing, as first observed by Aref [1]; see also Ottino [8]. Adapting this paradigm to the ocean—that is, scaling it up from laboratory fluids experiments to oceanic regions like the North Atlantic—is a major challenge.



Satellite image showing the Gulf Stream and its meanderings in the open North Atlantic. The image shows water surface temperature from AVHRR data for a time interval of 6.75 days, ending January 19, 1999. (Courtesy of the Johns Hopkins University Applied Physics Laboratory.)

The issue of scale is one of two fundamental issues that confront the mathematician who becomes interested in ocean dynamics. The other is experimentation. While there are highly ingenious and instructive laboratory experiments that simulate ocean activity, they cannot come close to reproducing the full gamut of effects present in even a restricted part of the ocean. Data collection is also fraught with difficulties of interpretation as the data are, of necessity, very sparse. Any measurements of subsurface activity, which is not accessible to satellite observations, involve sampling by floats that can render information only along the paths taken by the floats themselves. From the perspective of the mathematician, these issues only serve to make the subject fertile ground for investigation. The question of scales calls for innovative modeling of medium- and large-scale effects, and the difficulty with experimentation creates the need for good theories on which observational schemes can be based.

It is against this backdrop that a collaboration has developed between physical oceanographers at Woods Hole Oceanographic Institution (WHOI) and some of us in the Division of Applied Mathematics at Brown University. In 1992, when our collaboration began, Reza Malek-Madani of the U.S. Naval Academy was acting as an officer in the Applied Analysis Program at the Office of Naval Research. Envisioning a role for dynamical systems ideas in ocean studies, he encouraged two of his investigators, Steve Wiggins of Caltech and myself, to make contact with physical oceanographers at WHOI with a view to developing a project around the application of dynamical systems techniques to ocean transport.

The part that Malek-Madani played cannot be underestimated; it was a wonderful example of the creative role that a program officer can have as an instigator of new projects. Wiggins had worked on the application of dynamical systems techniques to fluid flows, particularly on what is known as "chaotic advection" [13]. He was also the adviser of Vered Rom-Kedar, whose pioneering work on lobe dynamics [10,11] forms the foundation for much of the work in this area. While I had experience with a variety of areas to which dynamical systems theory is applied, fluid motion was relatively new to me.

Wiggins and I visited WHOI in the fall of 1992. I then organized a workshop that was held the following spring. In attendance, among others, were the people from WHOI we had met the previous fall, including Joe Pedlosky, Larry Pratt, Roger Samelson, and Phil Richardson. The workshop was held at a beautiful inn near the ocean in southern Rhode Island. (I was told recently by a colleague that this is like talking of northern Monaco!) In this conducive environment, we developed a meeting format that promoted discussion and problem formulation. It seemed at the time as if little was achieved. I have since come to understand that such impressions can be deceptive, especially for a group that is striving to cross disciplinary boundaries. The sharing of ideas and expertise and the brainstorming on potential directions at this meeting were critical in setting both the agenda and the tone of future interactions.

We discussed a number of specific problems; the most pressing, and promising, was the use of the geometric techniques of dynamical systems to determine fluid exchange. The exchange of interest occurs on the edge of coherent structures, such as the northern or southern edges of the Gulf Stream, and it is through this exchange that the structures "communicate" with the ambient water. The dynamical systems perspective prescribes that the exchange is controlled by stable and unstable manifolds of distinguished (hyperbolic) trajectories.

A Project Comes into Focus

The program came fully to life in the fall of 1994. I had spent a sabbatical in Germany (using some of the time to learn basic geophysical fluid dynamics), and Pat Miller (now on the faculty at Stevens Institute of Technology) had started as a postdoc at Brown. Miller and I went to WHOI to talk with Larry Pratt, and it was during this visit that a project came into focus.

Pratt was working with Audrey Rogerson on the Lagrangian motion of a barotropic meandering jet. The model was numerical in that a pseudospectral method was used to solve the barotropic equations (the geophysical version of the two-dimensional Euler equations). The work of Flierl et al. [5] served as a guide for the creation of a sustained "numerical barotropic jet." We proceeded with the rather vague idea of using the geometric methods of dynamical systems in what we, the mathematicians, saw as a "real-life" problem.

I was to be quickly disabused of this preconception. What I saw as an applied, real-world problem was a "toy" problem to the oceanographers. Indeed, the numerical, barotropic jet was as close to an idealized model as it was possible to get, short of simply writing down an analytic model that exhibited the anticipated characteristics of a meandering jet. On the other hand, the oceanographers really did care about the outcome. The motivation for Larry Pratt's interest was that our work could give a definitive assessment of the importance of this fluid exchange process to the overall distribution of fluid around such coherent structures as a meandering jet, with the Gulf Stream as the canonical example.

The sharp potential vorticity (local fluid vorticity plus planetary vorticity) gradients in the Gulf Stream are known to inhibit cross-stream transport (see [2]). Sharp gradients of sea-surface temperature, which are closely correlated with potential vorticity, are evident in the dramatic satellite photograph of the Gulf Stream that appears on page 1. Water does, however, cross the Gulf Stream: Pieces of the current encircle patches of warm (or cold) water and pinch off to become so-called warm (cold) Gulf Stream rings to the north (south). One warm-core ring can be seen in the illustration pinching off at a peak of the Gulf Stream to the south of Nova Scotia. Based on subsurface float studies, Bower and Lozier [2] concluded that fluid does not appear to cross the Gulf Stream other than by ring pinch-off but that separation of fluid particles from the current on both sides is a common occurrence (observed at about 70% of the floats within 30 days).

It is the transport by which fluid particles leave the current that is explained by the dynamical systems techniques. Although not as dramatic as the bulk transport in ring pinch-off, this transport is more continual and prevalent. In accord with the observations, the water involved does not directly cross the Gulf Stream but rather moves between regions characterized by different types of motion, such as jet water, recirculating cells, and retrograde flow. Were the flow steady, these regions would be protected from each other by separating trajectories (separatrices). The series of cat's eyes (Stuart vortices) in a critical layer (see [4]) is a good example of this effect.

A more prosaic, and less relevant, example is the motion of a simple pendulum and the separatrix forming the boundary between the regimes of oscillations around the equilibrium position and rotations around the pivot. Fluctuations of the system over time will cause the separatrices to break down, facilitating transfer between these regions of different characteristic motion. For the pendulum, this would mean a switch from oscillation to rotation, or vice versa. For the fluid flow, the effect would be transport between regions of seemingly different types of motion—out of the jet (current) and into the ambient flow, for instance. Dynamical systems methods give us a way to explain this transport qualitatively and to estimate its extent quantitatively. The key idea is that the transport is orchestrated by stable and unstable manifolds that are residues of the separatrix after perturbation.

Adjustments in our Perspective

There were many obstacles to be overcome. As dynamicists, we were used to analyzing vector fields given to us as analytic expressions. We were now forced to accommodate vector fields in the form of numerical databases. Values of the field were given at discrete grid points; to supply values throughout the spatial domain, we needed to perform smart interpolations.

This problem had already been faced by computational fluid dynamicists, but we took an approach slightly different from the traditional one. Since our goal was intensive (Lagrangian) calculations to uncover potentially complex geometric structures involving stable and unstable manifolds, we solved the PDE just once, archived the result, and interpolated to fill in the data in the full domain. We then had, as input for a dynamical systems program, an algebraic (albeit very complicated) vector field. For these intensive Lagrangian calculations, this is a viable alternative to the standard procedure of simultaneously performing the Lagrangian calculation and numerically solving the PDE.

A more difficult challenge was to come from the fact that the data were available to us for only a finite time-span. The incredibly complex flows that arise in the ocean spawn coherent structures that last for only finite periods of time. An eddy that pinches off from a jet, such as a Gulf Stream ring, and is later re-entrained, for example, has a life-span on the order of months. Since most dynamical systems concepts are built around definitions that are asymptotic in time, we were forced to make a significant readjustment of our perspective.

As we soon realized, working with finite-time data-sets is inevitable for other reasons. In particular, a numerical model renders a velocity field that is defined only over a finite time interval; even if a structure looks persistent, concrete information about it will be available only for a fixed period of time and any extrapolation to an infinite time interval will be somewhat arbitrary. An approach that would acknowledge the finite-time nature of the data was clearly called for.

Such an approach was supplied first in the paper of Miller et al. [7]; see also [9]. The basic idea was to do what is done anyway with full data sets—namely, we would numerically compute the relevant geometric structures, using the finite-time data available. After all, it is possible to compute only over finite time intervals. In other words, operationally, we proceed exactly as if we were computing in the more traditional setting of an analytically prescribed vector field. In this way, we obtain "operational" hyperbolic points (points at centers of regions where compression and expansion occur) and attendant, "operational" stable and unstable manifolds. Since this work, a theoretical underpinning has been given to this finite-time transport by Haller and Poje [6], who assumed, quite reasonably, that the Eulerian time-scale is longer than the Lagrangian time-scale.

Promising Results

We applied this approach to the barotropic jet model with dramatic success. The structures emerged with great clarity. These "stable" and "unstable" manifolds can be used in a natural and effective way as a guide to the transport between regimes of ostensibly different motion, such as the retrograde flow north of the Gulf Stream and recirculating flows in the troughs. The areas involved can be calculated, and the results indicate that the transport associated with this mechanism is significant. Normalized to appropriate ocean parameters, it is a flow rate that is of the same order of magnitude as that associated with ring pinch-off: approximately 1–3 Sv. This is a considerable quantity of water: 1 Sv (Sverdrup) equals 10^6 m^3/sec, which can be thought of as roughly the combined flow rate of all the major rivers in the world.

After this project was under way, Wen Masters took over from Malek-Madani at ONR and, jointly with atmospheric scientists, started to develop a Departmental Research Initiative on the general theme of predictability. This initiative is the current context for our work, and once again a program officer at ONR has steered the work in a fruitful direction. The focus now, both in the joint work with Pratt and in a new collaboration with Albert Kirwan's group at Delaware, is to use the Lagrangian map of a flow that we obtain by constructing these invariant manifolds as a basis for the development of observational schemes.

Data acquisition through float studies is extremely important and presents many challenges. A good template of the flow field, one that delineates the Lagrangian fate of *different* parts of the flow field, would be extremely useful for interpreting the data. It could also serve as a guide for determining the most efficient float placement scheme. These floats are expensive (up to a few thousand dollars each), and our goal is to make prescriptions for smart placement so that the optimal information can be gained from the small number of floats that can be deployed. This is ongoing work, and many hurdles will have to be overcome before we will have what can realistically be seen as an effective processing tool for oceanographers, but preliminary results are promising.

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