Preface

Greetings to all! I always begin my classes with that exclamation, and since this book grew out of lecture notes for a real class with real students, let me begin the same way. The class is EE261: The Fourier Transform and Its Applications, at Stanford. It's considered a course for beginning graduate students and advanced undergraduates in engineering. I get a crowd from all over engineering and science, and even a few math people. As of my last count I've had the pleasure of sharing the subject with around 2,200 students. Not a huge number, but not a small one either.

I've worked on the notes over a number of years but I've been slow to publish them, not the least because there are many books for such a course, including some by friends and colleagues. However, I can honestly say that generations of students have found the notes to be helpful. I hear from them, and I have greatly benefitted from their suggestions. Many books are buffed and polished to an extent that the author sees only his own reflection. Not the case here. I hope that these pages still carry the voice of a teacher, enthusiastic and certainly unperfected. Me talking to you.

So here you go. If you want (virtual) live action, the lectures from one version of the course are available on YouTube through the Stanford Engineering Everywhere program. I've never watched them. The rest of this introduction is pretty much what I hand out to the students as a quick summary of what to expect.

The Fourier transform has it all. As a tool for applications it is used in virtually every area of engineering and science. In electrical engineering, methods based on the Fourier transform are used in all varieties of signal processing, from communications and circuit design to the analysis of imaging systems. In materials science, physics, chemistry, and molecular biology it is used to understand diffraction and the crystal structures of compounds. There's even a field called Fourier optics. In mathematics, Fourier series and the Fourier transform are cornerstones of the broad area known as harmonic analysis, and applications range from number theory to the modern formulation of the theory of partial differential equations. Lurking not so far beneath the surface are deep connections to groups and symmetry. Particularly widely used in practical applications is the discrete Fourier transform computed via the FFT (Fast Fourier Transform) algorithms. It is not an exaggeration to say that much of the increasingly digital world depends on the FFT.

Historically, Fourier analysis developed from employing sines and cosines to model periodic physical phenomena. This is the subject matter of Fourier series (our first topic), and here we learn that a complicated signal can be written as a sum of sinusoidal components, the simplest periodic signals.[1] Borrowing from musical terminology, where pure tones are single sinusoids and the frequency is the pitch, the component sinusoids are often called the harmonics. Complicated tones are sums of harmonics. In this way, with a periodic signal we associate a discrete set of frequencies — its spectrum — and the amount that each harmonic contributes to the total signal. If we know the spectrum and the amount that each harmonic contributes, then we know the signal, and vice versa. We analyze the signal into its component parts, and we synthesize the signal from its component parts.

The Fourier transform arose as a way of analyzing and synthesizing nonperiodic signals. The spectrum becomes a continuum of frequencies rather than a discrete set. Through the Fourier transform, and its inverse, we now understand that *every signal has a spectrum, and that the spectrum determines the signal.* This maxim surely ranks as one of the major secrets of the universe. A signal (often a time varying function, thus a representation in the "time domain") and its Fourier transform (a function depending on the spectrum, thus a representation in the "frequency domain") are equivalent in that one determines the other and we can pass back and forth between the two. The signal appears in different guises in the time domain and in the frequency domain, and this enhances the usefulness of both representations.

"Two representations for the same quantity" will be a steady refrain in our work. In signal processing, "filtering in the frequency domain" is an example of this, where operations are carried out on the spectrum to, in turn, produce a signal in the time domain having desired properties. Another important example of the use of the dual representations is the sampling theorem, which is fundamental in passing from an analog to a digital signal. In optics, examples are diffraction and interference phenomena; in physics, an example is the Heisenberg uncertainty principle. In mathematics, celebrated identities in number theory come from Rayleigh's identity, which, in physics, says that the energy of a signal can be computed in either the time domain or the frequency domain representation.

Underlying much of this development and its wide applicability is the notion of *linearity.* The operations of analyzing a signal into its component parts (taking the Fourier transform) and synthesizing a signal from its component parts (taking the inverse Fourier transform) are *linear operations*, namely integration. The principle of superposition applies to linear systems — the sum of inputs produces a sum of outputs — and one can thus work with a complicated signal by working with

[1] Electrical engineers speak in terms of signals and mathematicians in terms of functions. As far as I'm concerned, the two terms are interchangeable. I'll use both.

linear combinations, sums or integrals, of simpler signals. This is fundamental to all signal processing. The desire to extend the applicability of Fourier series and the Fourier transform led not only to an increasing array of real-world applications but also to purely mathematical questions. Investigations on many fronts led to a better understanding of approximation and of limiting processes, the nature of integration, linear operators, eigenfunctions, and orthogonality. In extending the range of the mathematical methods it became necessary to move beyond classical ideas about functions and to develop the theory and practice of distributions, also known as generalized functions. The need for such objects was recognized earlier by engineers for very practical applications and by physicists for computations in quantum mechanics.

In short, in this class there's a little magic every day. I want you to see that.

A few comments on audience and backgrounds. The majority of the students in my class are EEs but not everyone is, and collectively they all bring a great variety of preparation and interests. Students new to the subject are introduced to a set of tools and ideas that are used by engineers and scientists in an astonishing assortment of areas. Students who have seen bits and pieces of Fourier techniques in other courses benefit from a course that puts it all together. I frequently hear this from people who have taken the class. I urge the students to keep in mind the different backgrounds and interests and to keep an open mind to the variety of the material, while recognizing that with such a rich panorama we have to be quite selective.

The goals for the course are to gain a facility with *using* the Fourier transform, both specific techniques and general principles, and learning to recognize when, why, and how it is used. Together with a great variety, the subject also has a great coherence, and my hope is that students will appreciate both.

A few comments on math. This is a very mathematical subject, no doubt about it, and I am a mathematician, but this is not a mathematics textbook. What are the distinctions? One is the role of applications. It is true that much of the mathematical development of Fourier analysis has been independent of the growing list of applications in engineering and science, but the mathematicians are missing out on at least some of the fun. Moreover, the applications have proliferated largely because of the computational power now available, and the tilt toward computation is also viewed suspiciously by maybe a few (not all!) mathematicians. While we will be able to see only a fraction of the problems that call on Fourier analysis, for (almost) every mathematical idea we develop there will be significant applications. This is a feature not always found in courses and textbooks outside of engineering. For example, when speaking of applications in the preface to his well-known book *Fourier Integrals* (from 1937), the British mathematician E. C. Titchmarsh wrote:

> As exercises in the theory I have written out a few of these as it seemed to me that an analyst should. I have retained, as having a certain

> picturesqueness, some references to “heat”, “radiation”, and so forth; but the interest is purely analytical, and the reader need not know whether such things exist.

Bad attitude.

It is important, however, to at least touch on some of the mathematical issues that arise. Engineering, particularly electrical engineering, draws more and more on mathematical ideas and does so with increasing sophistication. Those ideas may have come from questions once considered to be solely in the realm of pure mathematics or may have had their start in real-world problems. This back-and-forth is not new, but the exchange between mathematical ideas and engineering or scientific applications has become more circuitous and sometimes strained. Consider this comment of the Russian mathematician V. I. Arnold in his graduate-level book on differential equations, *Geometric Methods in the Theory of Ordinary Differential Equations*:

> The axiomatization and algebraization of mathematics, after more than 50 years, has lead to the illegibility of such a large number of mathematical texts that the threat of complete loss of contact with physics and the natural sciences has been realized.

It can be hard for engineers, both students and working professionals, to open an advanced math book and get anywhere. That’s understandable, as per Arnold, but it is limiting. If you look at the current engineering and scientific literature where the Fourier transform is used, it looks pretty mathematical. To take one fairly current example, the subject of *wavelets* (which we won’t do) has become quite important in various parts of signal processing. So, too, *compressive sensing.* You’ll understand these and other new ideas much better if you know the mathematical infrastructure that we’ll develop. For a list of mathematical topics that we’ll call on see Appendix A.

You do need mathematics. You need to understand better the mathematics you’ve already learned, and you need the confidence to learn new things and then to use those new things. If you’re looking for a learning goal, as everyone seems to be doing, that’s a big one. Beyond the specific formulas, facts, etc., I hope that this book offers a dose of mathematical know-how, honest and helpful, and with a light touch.