Preface

We often observe *web-like* patterns of waves on the surface of shallow water. They are examples of nonlinear waves, and these patterns are generated by nonlinear interactions among several obliquely propagating solitary waves.

This is a book about those two-dimensional wave patterns and is based on a set of lectures I delivered May 20–24, 2013, as part of a series sponsored by the National Science Foundation (NSF) through the Conference Board on the Mathematical Sciences (CBMS). The title of the lecture series was "*Solitons in Two-Dimensional Water Waves and Applications to Tsunami*." The main purpose of the lectures was to introduce modern mathematical tools to analyze those wave patterns. These tools are from several mathematical areas including algebraic geometry, algebraic combinatorics, and representation theory. In the lectures, I tried to convince audiences (mainly young researchers and graduate students) that despite their abstract nature they are quite useful to gain a deeper understanding of two-dimensional wave interactions. The conference was organized by Ken-Ichi Maruno and Virgil Pierce, and it was held at the University of Texas at Pan American (now a part of the University of Texas Rio Grande Valley). In addition to my lectures, talks were given by M. Ablowitz, S. Chakravarty, P. Guyenne, A. Kasman, T. Mizumachi, H. Segur, L. Williams, and H. Yeh.

To begin with, let me start by quoting a well-known story of the first recognition of a solitary wave on a water surface. In August, 1834, John Scott Russell observed a large solitary wave in a shallow water channel in Scotland. He noted in his paper [122] on the subject that

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed ....*

This solitary wave is now known as an example of a *soliton* and is described by a solution of the Korteweg-de Vries (KdV) equation [82]. The KdV equation describes "one-dimensional" wave propagation such as beach waves parallel to the coast line or waves in a narrow canal, and is obtained in the leading order approximation of an asymptotic perturbation theory under the assumptions of weak nonlinearity (small amplitude) and weak dispersion (long waves). The KdV equation has a rich mathematical structure including the existence of solutions containing an arbitrary number of solitons, called N-soliton solutions, and the Lax pair, a pair of linear operators whose compatibility condition gives the KdV equation. The KdV equation is a prototype equation of the $1+1$ (space + time)-dimensional integrable systems. In particular, the initial value problem of the KdV equation has been extensively studied by means of the method of inverse scattering transform (IST) based on the Lax pair [49] (see also [97] by Miura for an excellent review of the KdV equation). It is well known that a general initial datum decaying rapidly in the spatial variable evolves to a number of individual solitons and weakly dispersive wave trains separate from the solitons. See, for example, [4, 106, 108, 146] for the review of the KdV equation and the related topics on the integrable systems in general.

In [61], Kadomtsev and Petviashvili proposed a $(2+1)$-dimensional dispersive wave equation to study the stability of the one-soliton solution of the KdV equation under the influence of weak transverse perturbations. This equation is now referred to as the KP equation. It turns out that the KP equation has a much richer structure than the KdV equation, and might be considered as the most fundamental integrable system in the sense that many known integrable systems can be derived as special reductions of the so-called KP hierarchy which consists of the KP equation together with its infinitely many symmetries (see, e.g., [108, 3, 36, 99]). The KP equation can be also represented in the Lax form; that is, there exists a pair of linear equations associated with an eigenvalue problem and an evolution of the eigenfunction, which enables the method of IST. However, unlike the case of the KdV equation, the IST for the KP equation does not seem to provide a practical method of solving the initial value problem for initial waves consisting of line-solitons in the far field (see, for example, [17, 18, 16, 144, 145]).

In two-dimensional wave phenomena in shallow water, we observe that wave refraction, reflection, and diffraction lead to obliquely interacting waves, and when their amplitudes are sufficiently large, nonlinear effects can have striking effects on the resulting surface patterns. In particular, the resonant interactions among these obliquely propagating solitary waves play a fundamental role in multidimensional wave phenomena. The original description of the soliton interaction for the KP equation was based on a two-soliton solution found in Hirota bilinear form (see, e.g., [57]), which has the shape of an "X," describing the intersection of two lines with an oblique angle and a phase shift at the intersection point *without* resonance. In [70] and this book, this X-shaped solution is sometimes referred to as the "O"-type soliton, where O stands for "original" or "ordinary." In his study of 1977 on an oblique interaction of two line-solitons, Miles [95, 96] pointed out that the O-type solution becomes singular if the angle of the intersection is smaller than a certain critical value depending on the amplitudes of the solitons. Miles then found that at the critical angle, the two line-solitons of the O-type solution interact *resonantly*, and a third wave is generated to form a "Y-shaped" wave as a result of three-wave resonant interaction. Indeed, it turns out that such Y-shaped resonant waves are exact solutions of the KP equation (see also [107, 111]). Miles applied his theory to study the Mach reflection phenomenon of an incident wave onto a vertical wall and predicted that the third wave, called the *Mach stem*, created by the resonant interaction can reach "four-fold" amplification of the incidence wave. Several laboratory and numerical experiments attempted to validate his prediction of four-fold amplification, but with no definitive success (see, for example, [48, 67, 138] for numerical experiments and [114, 94, 85] for laboratory experiments). The Mach reflection phenomenon is one of the main topics in this book, which will be discussed in some detail in Chapter 8.

After the discovery of the resonant phenomenon in the KP equation, several numerical and experimental studies were performed to investigate resonant interactions in other physical two-dimensional equations such as the ion-acoustic and magneto-hydrodynamic equations under the Boussinesq approximation (see, for example, [62, 63, 43, 103, 48, 138, 110, 142]). However, apart from these activities, no further progress has been made over almost 30 years in the study of the KP equation. It would appear that the general perception was that there were not many new and significant results left to be uncovered in the soliton theory of the KP equation.

Over the past several years, I have been working with several collaborators on the classification problem of the soliton solutions of the KP equation and their applications to shallow water waves. Our studies then have revealed a large variety of solutions that were totally overlooked in the past, and we found that some of those exact solutions are quite useful to study the Mach reflection problem [14, 70, 25, 24, 26, 75, 152, 85]. Our numerical studies [75, 65] indicate that the solution to the initial value problem of the KP equation with a certain class of initial waves converges asymptotically to some of these exact solutions, that is, a separation of dispersive radiation from the soliton solution similar to the case of the KdV soliton.

The main purpose of this book is to explain some details of these results on the solitary wave solutions of the KP equation, referred to as the *KP solitons*. There are eight chapters. The purpose of Chapter 1 is to discuss the derivation of the Boussinesq-type equation from the three-dimensional Euler equation for an irrotational and incompressible fluid under the assumptions of weak nonlinearity and weak dispersion using an asymptotic perturbation method. We then derive the KP equation from the Boussinesq-type equation under a further assumption with quasi-two-dimensionality. We also calculate the higher order corrections to the KP equation for the shallow water wave and discuss a normal form theory for the KP equation with higher order terms.

Chapter 2 presents the KP theory, where the main theme is to introduce the τ-*function*, which plays a key role in describing a large class of the KP solitons. We start to discuss the *Burgers* equation, which can be linearized to the diffusion equation. The Burgers equation describes weak shock waves in a dissipative medium. We then show an interesting connection between the KP equation and the Burgers equation and point out that the resonant interaction in the KP solitons can be explained by a confluence of the shock solutions of the Burgers equation. Extending the Burgers equation to a multicomponent system, we define the τ-function as a solution of the system. This extension will be identified as a part of the general theory of the KP equation developed by Sato (see [123, 124, 125, 126]). This chapter also includes a brief review of the Sato theory of the KP hierarchy, where Sato recognized that the solutions of the KP hierarchy could be written in terms of the orbits on an infinite-dimensional Grassmann variety (or Grassmannian). In this book, we mainly deal with a finite-dimensional version of the Sato theory.

Chapter 3 provides an elementary introduction to the real Grassmannian, denoted by $\mathrm{Gr}(N,M)$, the set of N-dimensional subspaces of $\mathbb{R}^M$. This gives a foundation for the classification problem of the KP solitons. We also discuss the Schubert decomposition of the Grassmannian and introduce the Young diagrams to parametrize the components of the decomposition, called the Schubert cells. We then provide a refinement of the Schubert decomposition given by a projection of the Deodhar decomposition which gives a refinement of the Bruhat decomposition of the flag variety. We introduce the *Go-diagram* to parametrize each component of the decomposition, which is a Young diagram filled with *black* and *white* stones in certain special ways. In particular, if the Go-diagram has only white stones, it gives the Ⅎ-*diagram* (or *Le-diagram*) introduced by Postnikov in [118] to parametrize the *totally nonnegative* Grassmannians, denoted by $\mathrm{Gr}(N,M)_{\geq 0}$. (Here Ⅎ is pronounced as "Le" and represents a special property of the arrangement of white stones; see also Section 3.6.2.) The total nonnegativity is necessary and sufficient for the *regularity* of the soliton solutions [77]. We then note that the Ⅎ-diagram can be parametrized by a permutation of the symmetric group. We also introduce several combinatorial tools such as the chord diagram to represent the permutation and a network representation of the Ⅎ-diagram to compute the element of $\mathrm{Gr}(N,M)_{\geq 0}$. In particular, the chord diagram provides a useful tool to describe the far-field structure of the KP solitons.

In Chapter 4, we present a classification theorem of the KP solitons which states that the τ-function from a point of $\mathrm{Gr}(N,M)_{\geq 0}$ generates a KP soliton that has asymptotically $M-N$ line-solitons for $y \ll 0$ and N line-solitons for $y \gg 0$. We call such a KP soliton an $(M-N,N)$-type soliton solution. Moreover, these solitons can be labeled by the derangements of the symmetric group, which parametrize the point of $\mathrm{Gr}(N,M)_{\geq 0}$. We also discuss some details of the special case of an (N,N)-type soliton solution whose asymptotic line-solitons are the same for both regions of $y \ll 0$ and $y \gg 0$. We refer to those KP solitons as *N-soliton solutions*. We then present the detailed structure of the KP solitons associated with the low-dimensional Grassmannians $\mathrm{Gr}(N,M)_{\geq 0}$. In particular, we discuss the cases with $M = 3$ and $M = 4$, which provide building blocks for the general KP solitons. The results in this chapter are useful for the remaining Chapters 6, 7, and 8.

In Chapter 5, we consider the soliton graph which is defined as a *tropical* limit of the contour plot of the KP soliton in the xy-plane for fixed t. The tropical limit means that we consider the variables (x,y,t) in a large scale for the contour plots of the KP solitons, so that each line-soliton can be approximated by a crest line of the soliton. Then the soliton graph forms a web-like structure consisting of piecewise connected line segments. We develop an algorithm to construct the soliton graphs for $t > 0$ and $t < 0$ based on the Ⅎ-diagrams associated with the KP soliton. We also consider the KP soliton including multitime variables of the KP hierarchy. Then we give a partial classification result for the soliton graphs in terms of the polyhedral structure of the multitime space. In particular, we show an interesting connection between the soliton graphs and the triangulations of a certain point set determined by a polygon inscribed in the parabola.

In Chapter 6, we discuss the stability problem for the KP solitons and also present some numerical simulations of the KP equation with a certain class of initial conditions which are somewhat close to exact solutions but not necessarily small perturbations. In particular, we study the stability problem of a KdV soliton, i.e., a one-soliton solution parallel to the y-axis, using an elementary perturbation argument and discuss the result in terms of the recent study by Mizumachi [100]. The main result is that for a small (amplitude) perturbation, one should observe a generation of a local phase shift which propagates along the crest of the KdV soliton. We also study numerically the interaction properties of line-solitons and show that the solution of the initial value problem with a certain class of initial waves asymptotically approaches some of those KP solitons discussed in Chapter 4. In particular, we propose a concept called “minimal completion,” which enables us to predict the KP soliton that the solution approaches, based on the chord diagram developed in Chapter 3.

Chapter 7 discusses the inverse problem in the sense that we construct a KP soliton which approximates a datum (wave pattern) observed in shallow water or a result obtained by numerical simulation. The inverse problem is to determine a τ-function from the data consisting of the amplitudes and slopes of the asymptotic line-solitons and the interaction pattern of line-solitons. That is, we identify a point of the totally nonnegative Grassmannian and the underlying vector space $\mathbb{R}^M$ from the wave data observed. One should note that we are not solving the inverse scattering problem based on the Lax pair for the KP equation. The inverse scattering method for arbitrary initial data is a wide-open problem for the KP equation.

In the final chapter, Chapter 8, we investigate some details of the Mach reflection phenomenon in terms of the normal form of the KP equation with higher order corrections discussed in Chapter 1. Here we provide historical background on the phenomenon which includes numerical simulations, water tank experiments, and the discrepancy between the results obtained by the Miles theory in [95, 96] and those experiments. We then re-examine his results in terms of the normal form theory of the KP equation with higher order corrections. The final goal is to show that the normal form theory improves the Miles theory in providing an excellent description of the Mach reflection phenomenon.

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