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Chapter 7

Matrix Recovery and Completion

In the preceding chapters, we have studied the problem of recovering an unknown *vector* from a limited number of linear measurements. Several sufficient conditions for robust sparse recovery have been proposed, including the robust null space property (RNSP) and the restricted isometry property (RIP). Methods for constructing measurements that satisfy the RIP, both deterministic as well as probabilistic, have been proposed. The object of study in the present chapter is the recovery of *low rank matrices* from a limited number of linear measurements. An important special case of matrix recovery is matrix *completion*, in which the measurements consist of specific components of the unknown matrix. The most widely used decoder map for matrix *recovery* is nuclear norm minimization. In the case of matrix *completion*, however, minimization of the max-norm (defined in Section 7.5 below) is also used as the decoder map.

A recent trend in the compressed sensing literature is to express a given matrix as a sum of two matrices, one of which is of low rank and the other is very sparse. This problem fits naturally into an area of research called "robust principal component analysis" (PCA). It can be solved by minimizing the sum of a nuclear norm and an ℓ_1-norm. This approach is discussed in Chapter 9.

For the most part, the emphasis in this chapter is on the recovery of *real* matrices $X \in \mathbb{R}^{n_r \times n_c}$. However, the contents of Section 7.1 are also applicable to *complex* matrices $X \in \mathbb{C}^{n_r \times n_c}$, so in that section it is assumed that $X \in \mathbb{C}^{n_r \times n_c}$. Throughout the chapter, it is assumed that $n_r \leq n_c$. This assumption leads to a reduction in notational clutter. Moreover, the assumption does not result in any loss of generality, because if the unknown matrix has more rows than columns, then the problem can be reformulated as one of recovering its transpose.

Therefore the most general problem under study in this chapter can be stated as follows: Suppose $X \in \mathbb{C}^{n_r \times n_c}$, and $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ is a linear map. The measurement vector y is given by $y = \mathcal{A}(X)$ if it is assumed that measurements are noise-free, and by $y = \mathcal{A}(X) + \eta$ if it is assumed that there is measurement noise. Specifically, let us assume that an upper bound ϵ for $\|\eta\|_2$ is known, which could be zero if $\eta = \mathbf{0}$. The estimate $\hat{X}$ is determined via

$$\hat{X} := \arg\min_{Z} \|Z\|_N \text{ s.t. } \|y - \mathcal{A}(Z)\|_2 \leq \epsilon, \tag{7.1}$$

or its analogue with $\|\cdot\|_N$ replaced by the max-norm $\|\cdot\|_M$. To quantify how good this estimate is, we recall the relevant parts of Definition 2.7. The above procedure is said to achieve **robust rank recovery of rank k** if there exist constants C and D such that

$$\|\hat{X} - X\|_F \leq C\theta_k(X, \|\cdot\|_N) + D\epsilon. \tag{7.2}$$

Note that, in contrast to the very general norms used in vector recovery, here the discussion is restricted to the Frobenius norm of the recovery error (which is the matrix analogue of the

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Euclidean norm) and the sparsity index of X with respect to the nuclear norm (which is the matrix analogue of the ℓ_1-norm). Also, (7.2) is used as the definition of robust rank recovery irrespective of the norm used in the decoding process (nuclear or max-norm).

7.1 ▪ Matrix Recovery via Vector Recovery

In this section we present a general approach for deriving sufficient conditions for matrix recovery on the basis of sufficient conditions for vector recovery. In Section 7.2, we define the rank-restricted isometry property (RRIP) and rank-robust null space property (RRNSP) and show that they are sufficient conditions for robust rank recovery. But for some cosmetic changes, the material is taken from [197].

We begin by introducing some notation. Throughout the section, we work with matrices in $\mathbb{C}^{n_r \times n_c}$ where $n_r \leq n_c$. For notational simplicity, let $n = \min\{n_r, n_c\} = n_r$. Given a vector $x \in \mathbb{C}^n$, let $\mathcal{D}(x) \in \mathbb{C}^{n\times n}$ denote the diagonal matrix whose diagonal is the vector x. If $X \in \mathbb{C}^{n_r \times n_c}$, then $\Sigma(X) \in \mathbb{R}_+^{n\times n}$ denotes the diagonal matrix of singular values of X, some of which could be zero if $\mathrm{rank}(X) < n$. Similarly, $\boldsymbol{\sigma}(X) \in \mathbb{R}_+^n$ denotes the singular value vector of X, that is, the diagonal of the matrix $\Sigma(X)$. Next, suppose $X \in \mathbb{C}^{n_r \times n_c}$. Then the singular value decomposition of X has the form $X = U\Sigma(X)V^\dagger$, where $U \in \mathbb{C}^{n\times n}, V \in \mathbb{C}^{n_c \times n}$, and $U^\dagger U = V^\dagger V = I_n$. We refer to U and V as a **unitary pair** if $U^\dagger U = V^\dagger V = I_n$. For a unitary pair (U, V), define

$$S(U, V) := \{U\mathcal{D}(x)V^\dagger : x \in \mathbb{C}^n\}. \tag{7.3}$$

Then $S(U, V)$ is an n-dimensional subspace of $\mathbb{C}^{n_r \times n_c}$ (where the dimension is over the field of complex numbers). Finally, suppose $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ is a linear operator, and that (U, V) is a unitary pair. Then a linear operator $A_{U,V} : \mathbb{C}^n \to \mathbb{C}^m$, which can also be thought of as a matrix $A_{U,V} \in \mathbb{C}^{m\times n}$, is said to be a **restriction** of the linear operator $\mathcal{A}$ if it is true that

$$A_{U,V}x = \mathcal{A}(U\mathcal{D}(x)V^\dagger),\ \forall x \in \mathbb{C}^n. \tag{7.4}$$

Next we define the notion of an "extension" property.

Definition 7.1. *Suppose P is a property of matrices in $\mathbb{C}^{m\times n}$. Then we say that P_e is an* ***extension property*** *of P if P_e is a statement about linear operators from $\mathbb{C}^{n_r \times n_c}$ to $\mathbb{C}^m$, such that, for every linear operator $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ and every unitary pair (U, V), the restriction matrix $A_{U,V}$ satisfies property P.*

A concrete illustration of an extension property is the rank-restricted isometry property (RRIP) introduced in Section 7.2.

Now we present two key lemmas.

Lemma 7.2. *Suppose $X, Y \in \mathbb{C}^{n_r \times n_c}$, and let $\boldsymbol{\sigma}(X), \boldsymbol{\sigma}(Y), \boldsymbol{\sigma}(X - Y)$ denote, respectively, the vectors of singular values of $X, Y, X - Y$. Then*

$$\|\boldsymbol{\sigma}(X) - \boldsymbol{\sigma}(Y)\|_1 \leq \|X - Y\|_N, \tag{7.5}$$

$$\|\boldsymbol{\sigma}(X) - \boldsymbol{\sigma}(Y)\|_1 \leq \|X + Y\|_N. \tag{7.6}$$

Proof. Recall the following result from Lemma 1.8: Let $n = \min\{n_r, n_c\}$, and let $\sigma_i(X)$, $\sigma_i(Y)$, $\sigma_i(X - Y)$, $i \in [n]$, denote the singular values of $X, Y, X - Y$, respectively. Then, for every $l \in [n]$, we have that

$$\sum_{i=1}^{l} |\sigma_i(X) - \sigma_i(Y)| \leq \sum_{i=1}^{l} \sigma_i(X - Y).$$

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If we substitute $l = n$ into the above equation, then we get

$$\sum_{i=1}^{n} |\sigma_i(X) - \sigma_i(Y)| \leq \sum_{i=1}^{n} \sigma_i(X - Y).$$

Now note that the left side is $\|\boldsymbol{\sigma}(X) - \boldsymbol{\sigma}(Y)\|_1$, while the right side equals $\|X - Y\|_N$ by (1.24). Therefore we have established (7.5). If we replace Y by $-Y$, then the singular values $\sigma_i(Y)$ remain unaffected. Therefore (7.5) implies (7.6).

The next lemma is the key to the results in this section.

Lemma 7.3. *Suppose $X, W \in \mathbb{C}^{n_r \times n_c}$ satisfy*

$$\|X + W\|_N \leq \|X\|_N. \tag{7.7}$$

Let $W = U\Sigma(W)V^\dagger$ be an SVD of W, and let $X_1 = -U\Sigma(X)V^\dagger$. Then

$$\|X_1 + W\|_N \leq \|X_1\|_N. \tag{7.8}$$

Remark: The point of the lemma is this: If $\|X + W\|_N \leq \|X\|_N$, then there exists another matrix X_1 *with the same unitary pair as W and the same nuclear norm as X* such that $\|X_1 + W\|_N \leq \|X_1\|_N$.

Proof. Note that if $x \in \mathbb{C}^n$ and (U, V) is a unitary pair, then

$$\|U\mathcal{D}(x)V^\dagger\|_N = \|x\|_1.$$

Therefore it follows that

$$\begin{aligned}\|X_1 + W\|_N &= \|U[-\Sigma(X) + \Sigma(W)]V^\dagger\|_N \\ &= \|\Sigma(W) - \Sigma(X)\|_N = \|\boldsymbol{\sigma}(W) - \boldsymbol{\sigma}(X)\|_1 \\ &\leq \|X + W\|_N \leq \|X\|_N = \|X_1\|_N.\end{aligned}$$

In the above chain of reasoning, we first use the fact that the nuclear norm is unitarily invariant, then (7.6), then (7.7), and again the fact that the nuclear norm is unitarily invariant.

To state the main theorem, we introduce the notion of one vector (or matrix) being "as good as" another in an optimization problem. First, for vector recovery, given $A \in \mathbb{C}^{m \times n}$, $y \in \mathbb{C}^m$, and $\epsilon \geq 0$, we say that $\hat{x} \in \mathbb{C}^n$ is **as good as** $x \in \mathbb{C}^n$ with respect to A, y, ϵ if

$$\|\hat{x}\|_1 \leq \|x\|_1, \|y - A\hat{x}\|_2 \leq \epsilon, \|y - Ax\|_2 \leq \epsilon. \tag{7.9}$$

The idea is that if we attempt to recover an unknown vector by solving the ℓ_1-norm optimization problem

$$\hat{z} = \arg\min_{z \in \mathbb{C}^n} \|z\|_1 \text{ s.t. } \|y - Az\|_2 \leq \epsilon, \tag{7.10}$$

then both x and $\hat{x}$ are feasible for this problem, and $\hat{x}$ is "as good as" x. In the same vein, given a linear map $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$, $y \in \mathbb{C}^m$, and $\epsilon \geq 0$, we say that $\hat{X} \in \mathbb{C}^{n_r \times n_c}$ is **as good as** $X \in \mathbb{C}^{n_r \times n_c}$ if

$$\|\hat{X}\|_N \leq \|X\|_N, \|y - \mathcal{A}(\hat{X})\|_2 \leq \epsilon, \|y - \mathcal{A}(X)\|_2 \leq \epsilon. \tag{7.11}$$

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The interpretation is similar to that in the vector recovery problem. If we attempt to recover an unknown matrix by solving the nuclear norm minimization problem

$$\hat{Z} = \underset{Z \in \mathbb{C}^{n_r \times n_c}}{\arg\min} \|Z\|_N \text{ s.t. } \|y - \mathcal{A}(Z)\|_2 \leq \epsilon, \tag{7.12}$$

then both $\hat{X}$ and X are feasible for this problem, and $\hat{X}$ is "as good as" X.

Now we state the main theorem of this section, for which purpose we introduce a few new symbols. Given a vector $x \in \mathbb{C}^n$, the symbol $x_\downarrow \in \mathbb{R}^n_+$ denotes the vector consisting of the magnitudes of the n components of x, arranged in nonascending order. Note that, while x can be a complex vector, $x_\downarrow$ is a real vector with nonnegative components. In the discussion below, $\|\cdot\|_v$ denotes any norm on $\mathbb{C}^n$ with the property that $\|x\|_v = \|x_\downarrow\|_v$. Clearly all ℓ_p-norms satisfy this condition. The matrix norm on $\mathbb{C}^{n_r \times n_c}$ (with $n_r \leq n_c$) corresponding to $\|\cdot\|_v$ is denoted by $\|\cdot\|_V$ and is defined by

$$\|X\|_V = \|\boldsymbol{\sigma}(X)\|_v.$$

It is clear from the definition that the matrix norm $\|\cdot\|_V$ is unitarily invariant. Moreover, if we choose $\|x\|_v = \|x\|_p$ for $p = 1, 2, \infty$, respectively, then the corresponding matrix norm $\|\cdot\|_V$ becomes the nuclear norm, the Frobenius norm, and the spectral norm, respectively. In principle we could choose other values of $p \in [1, \infty]$, and all these choices would generate valid unitarily invariant matrix norms, which are the Schatten p-norms.

The statement of the theorem is facilitated by the introduction of shorthand notation for four statements.

V1. A matrix A satisfies a property P.

V2. There exists a function $h : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $x \in \mathbb{C}^n$, $\eta \in \mathbb{C}^m$, where $\|\eta\|_2 \leq \epsilon$, $y = Ax + \eta$, and any $\hat{x}$ as good as x with respect to A, y, ϵ, we have that

$$\|\hat{x} - x\|_v \leq h(x_\downarrow, \epsilon). \tag{7.13}$$

M1. The linear operator $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ satisfies the extension property P_e, where $n_r \leq n_c$.

M2. There exists a function $h : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $X \in \mathbb{C}^{n_r \times n_c}$, $\eta \in \mathbb{C}^m$, where $\|\eta\|_2 \leq \epsilon$, $y = \mathcal{A}(X) + \eta$, and any $\hat{X}$ as good as X with respect to $\mathcal{A}, y, \epsilon$, we have that

$$\|\hat{X} - X\|_V \leq h(\boldsymbol{\sigma}(X), \epsilon). \tag{7.14}$$

Now we state the main theorem.

Theorem 7.4. *With these conventions, we have that*

$$(V1 \implies V2) \implies (M1 \implies M2). \tag{7.15}$$

Moreover, the function h in $M2$ can be taken as the function h in $V2$.

Proof. Suppose V1 $\implies$ V2 and that M1 is true; we wish to show that M2 is true. Accordingly, suppose that $X \in \mathbb{C}^{n_r \times n_c}$, $y = \mathcal{A}(X) + \eta$ where $\|\eta\|_2 \leq \epsilon$, and that $\hat{X}$ is as good as X with respect to $\mathcal{A}, y, \epsilon$. Thus

$$\|\hat{X}\|_N \leq \|X\|_N, \|y - \mathcal{A}(\hat{X})\|_2 \leq \epsilon, \|y - \mathcal{A}(X)\|_2 \leq \epsilon. \tag{7.16}$$

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Define $W = \hat{X} - X$, so that $\hat{X} = X + W$. It is now shown that

$$\|W\|_V \leq h(\boldsymbol{\sigma}(X), \epsilon). \tag{7.17}$$

By assumption, $\|X + W\|_N \leq \|X\|_N$. Suppose $W = U\Sigma(W)V^\dagger$ is an SVD of W, and as in Lemma 7.3, define $X_1 = -U\Sigma(X)V^\dagger$. Then $\|X_1 + W\|_N \leq \|X_1\|_N$. Now define $y_1 = \mathcal{A}(X_1) + \eta$. Then

$$\begin{aligned}\|\mathcal{A}(X_1 + W) - y_1\|_2 &= \|\mathcal{A}(W) - \eta\|_2 = \|\mathcal{A}(X + W) - \mathcal{A}(X) - \eta\|_2 \\ &= \|\mathcal{A}(\hat{X}) - y\|_2 \leq \epsilon,\end{aligned}$$

where the last step follows from (7.16).

Now let us set up a vector recovery problem, where the measurement matrix is $A_{U,V}$ (with U, V coming from the SVD of W), true vector $x_2 = -\boldsymbol{\sigma}(X)$, and $\hat{x}_2 = x_2 + w$, where $w = \boldsymbol{\sigma}(W)$. Therefore $\hat{x}_2 = -\boldsymbol{\sigma}(X) + \boldsymbol{\sigma}(W)$. Suppose $y_1 = \mathcal{A}(X_1) + \eta$ as defined above is the measured vector for this problem. Then the measurement error $\eta_1 = y_1 - A_{U,V}x_2$ is given by

$$\eta_1 = y_1 + A_{U,V}\boldsymbol{\sigma}(X) = \mathcal{A}(X_1) + \eta - \mathcal{A}(X_1) = \eta$$

because

$$A_{U,V}\boldsymbol{\sigma}(X) = \mathcal{A}(U\Sigma(X)V^\dagger) = -\mathcal{A}(X_1).$$

Therefore it follows that $\|y_1 - A_{U,V}x_2\|_2 \leq \epsilon$. Next,

$$\begin{aligned}A_{U,V}\hat{x}_2 &= A_{U,V}(-\boldsymbol{\sigma}(X) + \boldsymbol{\sigma}(W)) \\ &= \mathcal{A}(-U\Sigma(X)V^\dagger + U\Sigma(W)V^\dagger) = \mathcal{A}(X_1 + W).\end{aligned}$$

We have already shown that $\|y_1 - \mathcal{A}(X_1 + W)\|_2 \leq \epsilon$. Therefore

$$\|y_1 - A_{U,V}\hat{x}_2\|_2 \leq \epsilon.$$

Hence both $x_2, \hat{x}_2$ satisfy the constraint in (7.10) with y replaced by y_1, $\hat{x}$ replaced by $\hat{x}_2$, and x replaced by x. To show that $\hat{x}_2$ is as good as x_2 for this problem, it remains only to show that $\|\hat{x}_2\|_1 \leq \|x_2\|_1$. This is a ready consequence of Lemma 7.3. We have that

$$\begin{aligned}\|\hat{x}_2\|_1 &= \| - \boldsymbol{\sigma}(X) + \boldsymbol{\sigma}(W)\|_1 \\ &\leq \|X_1 + W\|_N \leq \|X_1\|_N = \|X\|_N = \|x\|_1\end{aligned}$$

because $X_1 = -U\Sigma(X)V^\dagger$. Therefore $\hat{x}_2$ is as good as x_2 with respect to $A_{U,V}, y_1, \epsilon$. By assumption V1 implies V2, which means that

$$\|\hat{x}_2 - x_2\|_v \leq h[(x_2)_\downarrow, \epsilon] = h(\boldsymbol{\sigma}(X), \epsilon), \tag{7.18}$$

because $\boldsymbol{\sigma}(X)$ is already in nonascending order. Recall now that $\hat{x}_2 = x_2 + w$, so that $\hat{x}_2 - x_2 = w = \boldsymbol{\sigma}(W)$. Therefore (7.18) implies that

$$\|\boldsymbol{\sigma}(W)\|_v \leq h(\boldsymbol{\sigma}(X), \epsilon). \tag{7.19}$$

Recall that $W = \hat{X} - X$. Now it follows from the definition of the matrix norm $\| \cdot \|_V$ that

$$\|\hat{X} - X\|_V = \|W\|_V = \|\boldsymbol{\sigma}(W)\|_v \leq h(\boldsymbol{\sigma}(X), \epsilon).$$

This is precisely (7.17).

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7.2 ▪ Matrix Recovery via Nuclear Norm Minimization

In this section, we present some conditions for matrix recovery via nuclear norm minimization. First we present conditions based on the null space of the measurement map; these conditions are analogous to the contents of Section 3.1. Then we present conditions based on the rank-restricted isometry property (RRIP); these conditions are analogous to the contents of Section 3.2.

7.2.1 ▪ Null Space–Based Properties

Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is a linear measurement map. As before, define its null space $\mathcal{N}(\mathcal{A})$ by

$$\mathcal{N}(\mathcal{A}) := \{Z \in \mathbb{R}^{n_r \times n_c} : \mathcal{A}(Z) = 0\}. \tag{7.20}$$

Throughout, let $\mathcal{M}(k)$ denote the subset of $\mathbb{R}^{n_r \times n_c}$ consisting of all matrices of rank k or less. Suppose without loss of generality that $n_r \leq n_c$, and let $n := \min\{n_r, n_c\} = n_r$. Suppose $X \in \mathbb{R}^{n_r \times n_c}$, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ denote the singular values of X. Then, as shown in (2.39), we have that

$$\theta_k(X, \|\cdot\|_N) := \min_{Z \in \mathbb{R}^{n_r \times n_c}} \|X - Z\|_N = \sum_{i=k+1}^{n} \sigma_i. \tag{7.21}$$

Thus the quantity $\theta_k(X, \|\cdot\|_N)$ is the matrix analogue of the k-sparsity index $\sigma_k(x, \|\cdot\|_1)$ for vectors. For convenience, let us also define

$$\bar{\theta}_k(X, \|\cdot\|_N) := \sum_{i=1}^{k} \sigma_i, \tag{7.22}$$

so that

$$\|X\|_N = \bar{\theta}_k(X, \|\cdot\|_N) + \theta_k(X, \|\cdot\|_N).$$

The next definition gives matrix analogues of Definitions 3.7, 3.10, and 3.14.

Definition 7.5. *A linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is said to satisfy the* ***rank-exact null space property (RENSP)*** *of order k if*

$$\bar{\theta}_k(Z, \|\cdot\|_N) < \theta_k(Z, \|\cdot\|_N), \ \forall Z \in \mathcal{N}(\mathcal{A}) \setminus \{0\}. \tag{7.23}$$

$\mathcal{A}$ is said to satisfy the ***rank-stable null space property (RSNSP)*** *of order k if there exists a constant $\rho \in (0, 1)$ such that*

$$\bar{\theta}_k(Z, \|\cdot\|_N) \leq \rho\theta_k(Z, \|\cdot\|_N), \ \forall Z \in \mathcal{N}(\mathcal{A}). \tag{7.24}$$

$\mathcal{A}$ is said to satisfy the ***rank-robust null space property (RRNSP)*** *of order k if there exist constants $\rho \in (0, 1)$ and $\tau \in \mathbb{R}_+$ such that*

$$\bar{\theta}_k(M, \|\cdot\|_N) \leq \rho\theta_k(M, \|\cdot\|_N) + \tau\|\mathcal{A}(M)\|, \ \forall M \in \mathbb{R}^{n_r \times n_c}. \tag{7.25}$$

As in the vector case, the RENSP and the RSNSP are required to hold for all $Z \in \mathcal{N}(\mathcal{A})$, whereas the RRNSP is required to hold for all $M \in \mathbb{R}^{n_r \times n_c}$.

In Definition 7.5 we wrote out $\theta_k(Z, \|\cdot\|_N)$ and $\bar{\theta}_k(Z, \|\cdot\|_N)$ for the sake of completeness. However, from here onwards we simply write $\theta_k(Z)$ and $\bar{\theta}_k(Z)$, because the nuclear norm is the only one for which we will need to compute these quantities.

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Next, we consider two decoder maps. If $y = \mathcal{A}(X)$ without any measurement noise, we define

$$\Delta(y) = \hat{X} := \underset{Z \in \mathbb{R}^{n_r \times n_c}}{\arg\min} \|A\|_N \text{ s.t. } y = \mathcal{A}(Z), \tag{7.26}$$

whereas if $y = \mathcal{A}(X) + \eta$ where $\|\eta\| \leq \epsilon$, we define

$$\Delta(y) = \hat{X} := \underset{Z \in \mathbb{R}^{n_r \times n_c}}{\arg\min} \|A\|_N \text{ s.t. } \|y - \mathcal{A}(Z)\| \leq \epsilon. \tag{7.27}$$

Next it is shown that, as in the case of vector recovery, the RENSP, RSNSP, and RRNSP, respectively, are sufficient conditions for exact-rank recovery, stable-rank recovery, and robust-rank recovery in the sense of Definition 2.7. In addition, the RENSP is also necessary for exact-rank recovery.

Throughout the proofs in the remainder of this subsection, we make use of Lemma 1.8, specifically (1.20), with $l = n$, which implies that

$$\sum_{i=1}^{n} |\sigma_i(X) - \sigma_i(Z)| \leq \sum_{i=1}^{n} \sigma_i(X - Z). \tag{7.28}$$

Now let $k \in [n]$, and rewrite (7.28) as

$$\begin{aligned} \|X - Z\|_N = \sum_{i=1}^{n} \sigma_i(X - Z) &\geq \sum_{i=1}^{n} |\sigma_i(X) - \sigma_i(Z)| \\ &\geq \sum_{i=1}^{k} [\sigma_i(X) - \sigma_i(Z)] + \sum_{i=k+1}^{n} [\sigma_i(Z) - \sigma_i(X)] \\ &= \bar{\theta}_k(X) - \bar{\theta}_k(Z) + \theta_k(Z) - \theta_k(X). \end{aligned} \tag{7.29}$$

Theorem 7.6 is a matrix analogue of Theorem 3.16.

Theorem 7.6. *Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is linear and satisfies the RRNSP of order k with constants ρ, τ. Define define $\Delta : \mathbb{R}^m \to \mathbb{R}^{n_r \times n_c}$ as in* (7.27)*. Then*

$$\|\hat{X} - X\|_N \leq 2\frac{1+\rho}{1-\rho}\theta_k(X) + \frac{4\tau}{1-\rho}\epsilon. \tag{7.30}$$

Proof. Note that $\|\hat{X}\|_N \leq \|X\|_N$.[26] Define $H = X - \hat{X}$ so that $\hat{X} = X - H$. Then applying (7.29) with Z replaced by H gives

$$\begin{aligned} \bar{\theta}_k(X) + \theta_k(X) = \|X\|_N \geq \|\hat{X}\|_N &= \|X - H\|_N \\ &\geq \bar{\theta}_k(X) - \bar{\theta}_k(H) + \theta_k(H) - \theta_k(X), \end{aligned}$$

or, after cancelling $\bar{\theta}_k(X)$ and rearranging,

$$\theta_k(H) - \bar{\theta}_k(H) \leq 2\theta_k(X). \tag{7.31}$$

Next, note that both X and $\hat{X}$ are feasible for the optimization problem in (7.27). Therefore

$$\|\mathcal{A}(H)\| = \|[\mathcal{A}(X) - y] - [\mathcal{A}(\hat{X}) - y]\| \leq \|\mathcal{A}(X) - y\| + \|\mathcal{A}(\hat{x}) - y\| \leq 2\epsilon.$$

[26] The proof does not actually use the fact that $\hat{X}$ is the *minimizer* in (7.27)—just the fact that $\|\hat{X}\|_N \leq \|X\|_N$.

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Applying the RRNSP to H gives

$$\bar{\theta}_k(H) \leq \rho\theta_k(H) + 2\tau\epsilon. \tag{7.32}$$

Inequalities (7.31) and (7.32) can be expressed as

$$\begin{bmatrix} 1 & -1 \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \theta_k(H) \\ \bar{\theta}_k(H) \end{bmatrix} \leq \begin{bmatrix} 2\theta_k(X) \\ 2\tau\epsilon \end{bmatrix}. \tag{7.33}$$

This resembles (3.35), and the remainder of the proof follows that of Theorem 3.16.

Theorem 7.7. *Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is linear and satisfies the RSNSP of order k with constant ρ. Define $\Delta : \mathbb{R}^m \to \mathbb{R}^{n_r \times n_c}$ as in* (7.26)*. Then*

$$\|\hat{X} - X\|_N \leq 2\frac{1+\rho}{1-\rho}\theta_k(X). \tag{7.34}$$

The proof is very similar to that of Theorem 7.6 and is left as an exercise for the reader.

Theorem 7.8, stated next, is the matrix analogue of Theorem 3.9.

Theorem 7.8. *Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is linear, and define $\Delta : \mathbb{R}^m \to \mathbb{R}^{n_r \times n_c}$ as in* (7.26)*. Then the following two statements are equivalent:*

1. *The pair (A, Δ) achieves exact-rank recovery of order k.*
2. *The map $\mathcal{A}$ satisfies the RENSP.*

Proof. 1 $\implies$ 2. Let $Z \in \mathcal{N}(\mathcal{A}) \setminus \{0\}$ be arbitrary, and let $Z = U\Sigma V^\top$ be an SVD of Z. Define $Z_1 = U\Sigma_1 V^\top$, $Z_2 = -U\Sigma_2 V^\top$, where

$$\Sigma_1 = \mathrm{Diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0), \Sigma_2 = \mathrm{Diag}(0, \ldots, 0, \sigma_{k+1}, \ldots, \sigma_n).$$

Then

$$Z = Z_1 - Z_2, \|Z_1\|_N = \bar{\theta}_k(Z), \|Z_2\|_N = \theta_k(Z).$$

Also, $Z \in \mathcal{N}(\mathcal{A}) \setminus \{0\}$ implies that $\mathcal{A}(Z_1) = \mathcal{A}(Z_2)$, and that $Z_2 \neq Z_1$. Now apply item 1 to Z_1. Then we must have that $\|Z_1\|_N < \|Z_2\|_N$. This holds for *every* $Z \in \mathcal{N}(\mathcal{A}) \setminus \{0\}$, which is item 2.

2 $\implies$ 1. Suppose $X \in \mathcal{M}(k)$, $S \in \mathbb{R}^{n_r \times n_c} \neq X$, and $\mathcal{A}(S) = \mathcal{A}(X)$. Suppose in addition that $\mathcal{A}$ satisfies the RENSP. We wish to show that $\|X\|_N < \|S\|_N$. Define $Z = X - S$ and note that $Z \in \mathcal{N}(\mathcal{A}) \setminus \{0\}$. Now apply (7.29) while noting that $\bar{\theta}_k(X) = \|X\|_N$ and $\theta_k(X) = 0$ because $X \in \mathcal{M}_k$. This gives

$$\|S\|_N = \|X - Z\|_N \geq \|X\|_N - \bar{\theta}_k Z + \theta_k(Z) > \|X\|_N.$$

This completes the proof.

7.2.2 ▪ Rank-Restricted Isometry Property

In this section we introduce the rank-restricted isometry property (RRIP) and show that it is an extension of the RIP in the sense of Definition 7.1. A direct consequence of this interpretation is that, for every result concerning the robust sparse recovery of vectors using ℓ_1-norm minimiza-

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tion, there is a corresponding result on the *rank* robust sparse recovery of matrices using nuclear norm minimization.

Definition 7.9. *A linear map $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ is said to satisfy the* ***rank-restricted isometry property (RRIP)*** *of rank k with constant δ if*

$$(1-\delta)\|X\|_F^2 \leq \|\mathcal{A}(X)\|_2^2 \leq (1+\delta)\|X\|_F^2, \ \forall X \in \mathcal{M}(k), \tag{7.35}$$

where $\mathcal{M}(k)$ denotes the set of matrices in $\mathbb{C}^{n_r \times n_c}$ of rank r or less.

Lemma 7.10. *The rank-restricted isometry property (RRIP) is an extension property of the restricted isometry property (RIP) in the sense of Definition 7.1.*

Proof. Recall from Definition 3.19 that a matrix A satisfies the RIP of order k with constant δ if

$$(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2, \ \forall x \in \Sigma_k. \tag{7.36}$$

To show that the RRIP is an extension property of the RIP, suppose that a linear operator $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ satisfies (7.35). The objective is to show that, for every unitary pair U, V, the restriction operator $A_{U,V}$ satisfies (7.36). Accordingly, let $x \in \Sigma_k$ be an arbitrary k-sparse vector, and let $X = U\mathcal{D}(x)V^\dagger$. Then the singular value vector $\boldsymbol{\sigma}(X)$ of the matrix X will be the same as the vector x, except for replacing each element of x by its modulus and permuting the elements to put them in nonincreasing order. Therefore it follows that $\|\boldsymbol{\sigma}(X)\|_2 = \|x\|_2$. Moreover, $\|X\|_F = \|\boldsymbol{\sigma}(X)\|_2 = \|x\|_2$. Finally, by the definition of the restriction matrix, we have that $\mathcal{A}(X) = A_{U,V}x$. Substituting these identities into (7.35) leads to

$$(1-\delta)\|x\|_2^2 \leq \|A_{U,V}x\|_2^2 \leq (1+\delta)\|x\|_2^2, \ \forall x \in \Sigma_k.$$

Therefore $A_{U,V}$ satisfies (7.36).

Now we show how Theorem 7.4 can be used to convert a vector recovery bound into a matrix recovery bound. Recall the following upper bounds derived in Theorem 3.20. In that theorem, the symbol $\hat{x}$ is defined via (3.52), that is, as the *minimizer* of $\|z\|_1$ subject to the constraint that $\|y - Az\|_2 \leq \epsilon$. However, an examination of the proof shows that this fact is never used—only the fact that $\|\hat{x}\|_1 \leq \|x\|_1$ is used. Therefore, while proving Theorem 3.20, we have actually proved the following set of bounds.

Lemma 7.11. *Suppose that, for some number $t > 1$, the matrix $A \in \mathbb{C}^{m \times n}$ satisfies the RIP of order tk with constant $\delta_{tk} =: \delta < \sqrt{(t-1)/t}$. Define constants a, b, c, ρ, τ as in* (3.49) *and* (3.50)*. Suppose $x \in \mathbb{C}^n$ and that $y = Ax + \eta$ where $\|\eta\|_2 \leq \epsilon$. Choose any $\hat{x} \in \mathbb{C}^r$ such that*

$$\|\hat{x}\|_1 \leq \|x\|_1, \|y - A\hat{x}\|_2 \leq \epsilon. \tag{7.37}$$

Then

$$\|\hat{x} - x\|_1 \leq \frac{2(1+\rho)}{1-\rho}\sigma_k(x, \|\cdot\|_1) + \frac{4\tau}{1-\rho}\epsilon. \tag{7.38}$$

For all $p \in (1, 2]$ we have

$$\|\hat{x} - x\|_p \leq \frac{1}{k^{1-1/p}} \cdot \frac{2}{1-\rho}[(1+2\rho)\sigma_k(x, \|\cdot\|_1) + 3\tau\epsilon]. \tag{7.39}$$

In particular,

$$\|\hat{x} - x\|_2 \leq \frac{2}{\sqrt{k}(1-\rho)}[(1+2\rho)\sigma_k(x, \|\cdot\|_1) + 3\tau\epsilon]. \tag{7.40}$$

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By combining Theorem 7.4 with Lemmas 7.10 and 7.11, we get the following set of bounds for matrix recovery. Note that, if we choose the vector norm $\|\cdot\|_v$ in (7.13) as the ℓ_1-norm, then the corresponding matrix norm $\|\cdot\|_V$ in (7.14) becomes the nuclear norm. If we choose $\|\cdot\|_v$ as the ℓ_2-norm, then $\|\cdot\|_V$ becomes the Frobenius norm.

Theorem 7.12. *Suppose that, for some number $t > 1$, the linear map $\mathcal{A} : \mathbb{C}^{n_r \times n_c} \to \mathbb{C}^m$ satisfies the RRIP of order tk with constant $\delta_{tk} < \sqrt{(t-1)/t}$. Define constants a, b, c, ρ, τ as in* (3.49) *and* (3.50)*. Suppose $X \in \mathbb{C}^{n_r \times n_c}$ and that $y = \mathcal{A}(X) + \eta$ where $\|\eta\|_2 \leq \epsilon$. Choose any $\hat{X} \in \mathbb{C}^{n_r \times n_c}$ such that*

$$\|\hat{X}\|_N \leq \|X\|_N, \|y - \mathcal{A}(\hat{X})\|_2 \leq \epsilon. \tag{7.41}$$

Then

$$\|\hat{X} - X\|_N \leq \frac{2(1+\rho)}{1-\rho}\theta_k(X, \|\cdot\|_N) + \frac{4\tau}{1-\rho}\epsilon, \tag{7.42}$$

$$\|\hat{X} - X\|_F \leq \frac{2}{\sqrt{k}(1-\rho)}[(1+2\rho)\theta_k(X, \|\cdot\|_N) + 3\tau\epsilon], \tag{7.43}$$

where the rank sparsity indices θ are defined in (2.41) *and* (2.40)*, respectively. In particular, if $\hat{X}$ is determined as*

$$\hat{X} = \arg\min_Z \|Z\|_N \text{ s.t. } \|y - \mathcal{A}(\hat{X})\|_2 \leq \epsilon, \tag{7.44}$$

then $\hat{X}$ satisfies (7.42) *and* (7.43)*.*

In Theorem 3.24, it is shown that the bound $\delta_{tk} < \sqrt{(t-1)/t}$ is tight. An analogous statement is also true for matrix recovery.

Theorem 7.13. *(See [49, Proposition 3.2].) Suppose $t \geq 4/3$ and is rational, and $\epsilon > 0$ is arbitrarily small. There exist integers r and n and a linear map $\mathcal{A}$ on $\mathbb{R}^{n \times n}$ such that the following hold:*

1. *$\mathcal{A}$ satisfies the RRIP of rank tr with*

$$\delta_{tr} \leq \sqrt{\frac{t-1}{t}} + \epsilon.$$

2. *There exists a matrix X of rank r that cannot be recovered using* (7.26)*.*

It is stated (without proof) in [251, Remark 3] that there is a matrix analogue of Theorem 3.25.

Problem 7.1. *This is an analogue of Lemma 3.8 for matrices. Using the fact that $\|Z\|_N = \theta_k(Z, \|\cdot\|_N) + \bar{\theta}_k(Z, \|\cdot\|_N)$ for all Z, show that the following statements are equivalent:*

1. *A linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ satisfies the RENSP.*
2. $\bar{\theta}_k(Z, \|\cdot\|_N) < (1/2)\|Z\|_N, \ \forall Z \in \mathcal{N}(A)$.
3. $\|Z\|_N < 2\theta_k(Z, \|\cdot\|_N), \ \forall Z \in \mathcal{N}(A)$.

Problem 7.2. *This is an analogue of Lemma 3.11 for matrices. Show that the following statements are equivalent:*

1. *A linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ satisfies the RSNSP with constant $\rho \in (0, 1)$.*
2. $\bar{\theta}_k(Z, \|\cdot\|_N) < (\rho/(1+\rho))\|Z\|_N, \ \forall Z \in \mathcal{N}(A)$.
3. $\|Z\|_N < (1+\rho)\theta_k(Z, \|\cdot\|_N), \ \forall Z \in \mathcal{N}(A)$.

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7.2.3 ▪ The RRIP Implies the RRSNP

In the previous subsection, the fact that the RRIP leads to robust-rank recovery was derived as a consequence of Theorem 7.4. In the present subsection, we present an alternate derivation by showing that the RRIP implies the RRNSP and then invoking Theorem 7.6.[27]

Theorem 7.14. *Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ satisfies the RRIP of rank tk with $\delta_{tk} < \sqrt{(t-1)/t}$ for some $t > 1$. Define the constants ν, a, b, c as in* (3.48) *and* (3.49) *and the constants ρ, τ as in* (3.50)*. Then, $\mathcal{A}$ satisfies the RRNSP of rank k with constants ρ and τ.*

To aid in the proof of this theorem, we present a series of lemmas. The first lemma is just a specialization of Lemma 7.3 to real matrices.

Lemma 7.15. *Given $W \in \mathbb{R}^{n_r \times n_c}$ with the singular value decomposition $U\Sigma(W)V^\top$, suppose there exists an $X_0 \in \mathbb{R}^{n_r \times n_c}$ for which $\|X_0 + W\|_N \leq \|X_0\|_N$. Then, for $X_1 = -U\Sigma(X_0)V^\top$, we have that*

$$\|X_1 + W\|_N \leq \|X_1\|_N.$$

Lemma 7.16. *Suppose $U \in \mathbb{R}^{n_r \times n_r}, V \in \mathbb{R}^{n_c \times n_r}$ form a unitary pair. Let $x \in \mathbb{R}^{n_r}$ be arbitrary. Define $A \in \mathbb{R}^{n_r \times n_c}$ as*

$$A = UD(x)V^\top.$$

Then the following statements are true:

1. $\operatorname{rank}(A) = \|x\|_0$*, where $\|x\|_0$ is the cardinality of the support set of x.*
2. *A has the singular values $|x_1|,\ |x_2|, \ldots, |x_{n_r}|$ (not necessarily in decreasing order).*

Proof. Define a diagonal matrix $I^* \in \mathbb{R}^{n_r \times n_r}$ as

$$(I^*)_{i,i} = \begin{cases} -1, & \text{for } x_i < 0, \\ 1, & \text{for } x_i \geq 0. \end{cases}$$

Then ${I^*}^\top = I^*$, ${I^*}^\top I = I{I^*}^\top = I^*$, and $I^* I^* = I$. Moreover, we can write

$$A = UI^* D(|x|)V^\top.$$

Now UI^* is also unitary. Therefore the above factorization looks like a singular value decomposition but for the fact that the elements of $|x|$ might not be in nonincreasing order. But this can be sorted out by row and column permutations. This shows that the components of $|x|$ are the singular values of A.

Lemma 7.17. *Let $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ be a linear measurement map and let (U, V) be a unitary pair with $U \in \mathbb{R}^{n_r \times n_r}$, $V \in \mathbb{R}^{n_c \times n_r}$. Suppose the two sets of vectors $\{u_1, u_2, \ldots, u_{n_r}\}$, $\{v_1, v_2, \ldots, v_{n_r}\}$ represent the columns of U and V, respectively. Define a matrix $A_{UV} \in \mathbb{R}^{m \times n_r}$ as*

$$A_{UV} = \left[\mathcal{A}(u_1 v_1^\top) | \mathcal{A}(u_2 v_2^\top) | \ldots | \mathcal{A}(u_{n_r} v_{n_r}^\top)\right]. \tag{7.45}$$

Let $x \in \mathbb{R}^{n_r}$ be arbitrary. Then

$$\mathcal{A}(UD(x)V^\top) = A_{UV}x.$$

[27] This proof is by Shashank Ranjan; see [204].

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Proof. We have

$$\mathcal{A}(UD(x)V^\top) = \mathcal{A}\Big(\sum_{i=1}^{n_r} x_i u_i v_i^\top\Big). \tag{7.46}$$

Using the linearity of $\mathcal{A}$, (7.46) can be written as

$$\begin{aligned}\mathcal{A}(UD(x)V^\top) &= \sum_{i=1}^{n_r} x_i \mathcal{A}(u_i v_i^\top)\\ &= \big[\mathcal{A}(u_1 v_1^\top)|\mathcal{A}(u_2 v_2^\top)|\dots|\mathcal{A}(u_{n_r} v_{n_r}^\top)\big]x\\ &= A_{UV}x.\end{aligned}$$

Lemma 7.18. *Suppose $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ satisfies the RRIP of rank k with $\delta_k := \delta$. Then, for a given unitary pair (U, V), the matrix A_{UV} (as defined in* (7.45)*) satisfies the RIP of order k with $\delta_k := \delta$.*

Proof. Let $x \in \Sigma_k \subseteq \mathbb{R}^{n_r}$ be arbitrary. Define

$$X = UD(x)V^\top.$$

From Lemma 7.16 , $\operatorname{rank}(X) = \|x\|_0 \le k$, and X has the singular values $\{|x_1|, |x_2|, \dots, |x_{n_1}|\}$, which implies that $X \in \mathcal{M}_k$. Now we make the use of Lemma 7.17 and the RRIP of $\mathcal{A}$, namely

$$(1-\delta)\|X\|_F^2 \le \|\mathcal{A}(X)\|_2^2 = \|A_{UV}x\|_2^2 \le (1+\delta)\|X\|_F^2. \tag{7.47}$$

Observe that

$$\|X\|_F^2 = \sum_{i=1}^{n_r} \sigma_i^2(X) = \sum_{i=1}^{n_r} |x_i|^2 = \|x\|_2^2.$$

Hence we can rewrite (7.47) as

$$(1-\delta)\|x\|_2^2 \le \|A_{UV}x\|_2^2 \le (1+\delta)\|x\|_2^2. \tag{7.48}$$

This completes the proof.

Now we present the proof the main theorem of this subsection.

***Proof of Theorem* 7.14.** Let $X \in \mathbb{R}^{n_r \times n_c}$ be arbitrary and let $X = U\Sigma(X)V^\top$ be the singular value decomposition of X. Define A_{UV} as in (7.46), and define $S = [k]$.

From Lemma 7.18, it follows that A_{UV} satisfies the RIP of order tk with $\delta_{tk} < \sqrt{(t-1)/t}$. Consequently, from Theorem 3.20, A_{UV} satisfies the RNSP of order k with constants ρ, τ defined as above. Now let $\boldsymbol{\sigma}(X)$ denote the vector of singular values of X. Then $\mathcal{A}(X) = A_{UV}\boldsymbol{\sigma}(X)$. Using Definition 3.14, we get

$$\|\boldsymbol{\sigma}(X)_S\|_1 \le \rho\|\boldsymbol{\sigma}(X)_{S^c}\|_1 + \tau\|A_{UV}\boldsymbol{\sigma}(X)\|_2. \tag{7.49}$$

Using the fact that $\|\boldsymbol{\sigma}(X)_S\|_1 = \bar{\theta}_k(X, \|\cdot\|_N)$, $\|\boldsymbol{\sigma}(X)_{S^c}\|_1 = \theta_k(X, \|\cdot\|_N)$, and $\|A_{UV}\boldsymbol{\sigma}(X)\|_2 = \|\mathcal{A}(X)\|_2$ (from Lemma 7.18), the inequality (7.49) can be written as

$$\bar{\theta}_k(X, \|\cdot\|_N) \le \rho\theta_k(X, \|\cdot\|_N) + \tau\|\mathcal{A}(X)\|_2, \tag{7.50}$$

which is the desired inequality.

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7.3 ▪ Probabilistic Recovery Methods

Theorem 7.12 shows clearly the importance of constructing linear maps $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ that satisfy the RRIP. In the case of vector recovery, we have seen that if the measurement matrix consists of random samples of a sub-Gaussian variable (together with the normalization factor $(1\sqrt{m})$), then the resulting matrix satisfies the RIP with high probability. In this section, we state and prove an analogue of this result for linear maps on the set of matrices. Interestingly, as of now there are no *deterministic* procedures for constructing measurement maps that satisfy the RRIP. It is evident that any linear map $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ can be represented as a matrix of dimensions $m \times n_r n_c$, just by representing each matrix $X \in \mathbb{R}^{n_r \times n_c}$ by a vector $v(X) \in \mathbb{R}^{n_r n_c}$. The difficulty however is that the restriction on the rank of X translates into highly nonlinear constraints on the components of $v(X)$.

Suppose we wish to recover an unknown matrix $X \in \mathbb{R}^{n_r \times n_c}$ using m linear measurements. If $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ is a linear measurement map, then we can represent $\mathcal{A}$ as follows:

$$\mathcal{A}(X) = \begin{bmatrix} \langle A_1, X \rangle_F \\ \vdots \\ \langle A_m, X \rangle_F \end{bmatrix}, \tag{7.51}$$

where $A_1, \ldots, A_m \in \mathbb{R}^{n_r \times n_c}$. Now suppose that each matrix A_l equals $(1/\sqrt{m})\Phi_l$, where Φ_l consists of $n_r n_c$ independent samples of a zero-mean, unit-variance random variable Y, which is sub-Gaussian in the sense of Definition 6.12. Thus there exists a constant $c > 0$ such that

$$E[\exp(\theta Y)] \leq \exp(c\theta^2), \ \forall \theta \in \mathbb{R}. \tag{7.52}$$

The objective in this section can be stated as follows: Given a rank bound k, an RRIP constant δ, and a failure probability ξ, determine a lower bound on the number of measurements m such that the map $\mathcal{A}$ satisfies the RRIP of order k with constant δ, with probability $\geq 1 - \xi$. The principal results of this section are Theorems 7.21 and 7.22. For the most part, the contents of this section mirror those of [56]. However, the actual proof of the RRIP is different and parallels the proof of the RIP in [19]; it is presented here in Chapter 6.

For a random variable Y that satisfies (7.52), we can define a corresponding constant $\tilde{c}$ as in (6.43), namely

$$\gamma = 2, \zeta = 1/(4c), \alpha = \gamma e^{-\zeta} + e^{\zeta}, \beta = \zeta, \tilde{c} := \frac{\beta^2}{2(2\alpha + \beta)}. \tag{7.53}$$

With these definitions, we can state the following analogue of Theorem 6.17.

Theorem 7.19. *Suppose matrices $\Phi_1, \ldots, \Phi_m$ each consist of $n_r n_c$ independent samples of a sub-Gaussian random variable Y that satisfies* (7.52). *Define $A_l = (1/\sqrt{m})\Phi_l$, and define $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ as in* (7.51). *Suppose $X \in \mathbb{R}^{n_r \times n_c}$ satisfies $\|X\|_F = 1$. Then*

$$\Pr\{|\|\mathcal{A}(X)\|_2^2 - 1| > t\} \leq 2\exp(-\tilde{c}mt^2), \tag{7.54}$$

where $\tilde{c}$ is defined in (7.53).

The proof is the same as that of Theorem 6.17. Observe that if $A \in \mathbb{R}^{m \times n}$ and $u \in \mathbb{R}^n$, then

$$Au = \begin{bmatrix} a^1 u \\ \vdots \\ a^m u \end{bmatrix}, \|Au\|_2^2 = \sum_{i \in [m]} (a^i u)^2.$$

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If $X \in \mathbb{R}^{n_r \times n_c}$ and $\mathcal{A}(X)$ is given by (7.51), then

$$\|\mathcal{A}(X)\|_2^2 = \sum_{i \in [m]} (\langle A_i, X \rangle_F)^2.$$

So the same reasoning applies as in the proof of Theorem 6.17.

Theorem 7.19 gives a tail probability estimate for *one fixed* matrix X of unit Frobenius norm. However, what we need is a bound that is *uniform* over the set

$$\mathcal{S}(k) := \{X \in \mathcal{M}(k) : \|X\|_F = 1\},$$

where as before $\mathcal{M}(k)$ denotes the set of all matrices that have rank $\leq k$. For this purpose we follow the same general approach as in Section 6.2. Specifically, we compute the covering number of the set $\mathcal{S}(k)$; however, the details are a little different from those in Section 6.2. We begin by estimating the covering number of the set $\mathcal{S}(k)$ with respect to the Frobenius norm.

Theorem 7.20. *Let $N(\epsilon)$ denote the covering number of $\mathcal{S}(k)$ with respect to $\|\cdot\|_F$. Then*

$$N(\epsilon) \leq \left(1 + \frac{6}{\epsilon}\right)^{(n_r+n_c+1)k}. \tag{7.55}$$

Remark: The method of proof below is taken from that of [56, Lemma 3.1]. However, the actual estimate given here is slightly better.

Proof. Clearly $\mathcal{S}(k)$ consists of all matrices of the form $U\Sigma V^\top$, where $U \in \mathbb{R}^{n_r \times k}$, $U^\top U = I_k$, $V \in \mathbb{R}^{n_c \times k}$, and $V^\top V = I_k$, and $\Sigma = \mathrm{Diag}(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} \in \mathbb{R}_+^k$ and $\|\boldsymbol{\sigma}\|_2^2 = 1$. The approach is to construct an $\epsilon/3$ cover for each set (of U, V, and Σ).

We begin with the set $\{\boldsymbol{\sigma} \in \mathbb{R}_+^k : \|\boldsymbol{\sigma}\|_2^2 = 1\}$. This is a subset of the unit sphere (not the entire unit ball) in $\mathbb{R}^k$. Therefore, by Lemma 6.3, with respect to $\|\cdot\|_2$, this set has an $\epsilon/3$-cover of cardinality $\leq (1 + (6/\epsilon))^k$. Next, let us look at the set $\{U \in \mathbb{R}^{n_r \times k} : U^\top U = I_k\}$. Observe that

$$\|U\|_{1\to 2} = \max_{j \in [k]} \|u_j\|_2 = 1,$$

because each column of U is normalized. Now the unit sphere in the norm $\|\cdot\|_{1\to 2}$ is contained in the k-fold Cartesian product of the unit sphere in $\|\cdot\|_2$ in $\mathbb{R}^{n_r}$. By Lemma 6.3, this unit sphere has an $\epsilon/3$-cover of cardinality $\leq (1 + (6/\epsilon))^{n_r}$, whence the unit sphere in the norm $\|\cdot\|_{1\to 2}$ has an $\epsilon/3$-cover of cardinality $\leq (1+(6/\epsilon))^{n_r k}$. Now the set $\{U \in \mathbb{R}^{n_r \times k} : U^\top U = I_k\}$ is a subset of this latter unit sphere. Hence this set also has an $\epsilon/3$-cover of cardinality $\leq (1+(6/\epsilon))^{n_r k}$. By parallel reasoning, $\{V \in \mathbb{R}^{n_c \times k} : V^\top V = I_k\}$ has an $\epsilon/3$-cover of cardinality $\leq (1+(6/\epsilon))^{n_c k}$.

Now we use these to construct an ϵ-cover for $\mathcal{S}(k)$. The claim is that the set of all products of the form $U_i \Sigma_j V_l^\top$ as U_i, Σ_j, V_l range over their respective covers forms an ϵ-cover for $\mathcal{S}(k)$. The cardinality of the set of all such triple products is bounded by the right side of (7.55). Now it is shown that the set of all such products is indeed an ϵ-cover for $\mathcal{S}(k)$. Suppose $X = U\Sigma V^\top \in \mathcal{S}(k)$, and choose U_1, Σ_1, V_1 such that

$$\|U - U_1\|_{1\to 2} \leq \epsilon/3, \|\Sigma - \Sigma_1\|_F = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_1\|_2 \leq \epsilon/3, \|V - V_1\|_{1\to 2} \leq \epsilon/3,$$

and define $X_1 = U_1 \Sigma_1 V_1^\top$. Using the identity

$$abc - a_1 b_1 c_1 = (a - a_1)bc + a_1(b - b_1)c + a_1 b_1 (c - c_1),$$

we can write

$$\|X - X_1\|_F \leq \|(U - U_1)\Sigma V^\top\|_F + \|U_1(\Sigma - \Sigma_1)V^\top\|_F + \|U_1 \Sigma_1 (V - V_1)^\top\|_F.$$

We now bound each term separately.

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For the first term, note that right multiplication by $V^\top$ preserves the Frobenius norm because $V^\top V = I_k$. Therefore

$$\|(U - U_1)\Sigma V^\top\|_F = \|(U - U_1)\Sigma\|_F.$$

Next

$$\|(U - U_1)\Sigma\|_F^2 = \sum_{j\in[k]} \sigma_j^2 \|u_j - (u_1)_j\|_2^2 \leq \sum_{j\in[k]} \sigma_j^2 \|U - U_1\|_{1\to 2}^2 \leq (\epsilon/3)^2.$$

Hence $\|(U - U_1)\Sigma V^\top\|_F \leq \epsilon/3$. By parallel reasoning $\|U_1\Sigma_1(V - V_1)^\top\|_F \leq \epsilon/3$. As for the middle term,

$$\|U_1(\Sigma - \Sigma_1)V^\top\|_F = \|\Sigma - \Sigma_1\|_F = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_1\|_2 \leq \epsilon/3.$$

Combining these bounds shows that $\|X - X_1\|_F \leq \epsilon$.

Now suppose an RRIP constant $\delta \in (0, 1)$ and a "failure probability" $\xi \in (0, 1)$ are specified. Using the bound (7.55), we can derive a bound on the number of measurements m to ensure that $\mathcal{A}$ of (7.51) satisfies the RRIP of order k with constant δ, with probability $\geq 1 - \xi$.

Theorem 7.21. *Define $\mathcal{A} : \mathbb{R}^{n_r\times n_c} \to \mathbb{R}^m$ as in Theorem 7.19, and define $\tilde{c}$ as in* (7.53)*. If*

$$m \geq \frac{1}{\tilde{c}\delta^2}\left[7k(n_r + n_c + 1) + \frac{5}{4}\ln\frac{2}{\xi}\right], \tag{7.56}$$

then $\mathcal{A}$ of (7.51) *satisfies the RRIP of order k with constant δ, with probability $\geq 1 - \xi$.*

Remark: The proof of [56, Theorem 2.3] proceeds along somewhat different lines, and the resulting bound for the number of samples m is not so explicit as in (7.56). Instead, the proof given here is an adaptation of that of Theorem 6.1, which is taken from [19].

Proof. Define the adjoint operator $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{R}^{n_r\times n_c}$ in the standard manner, namely

$$\langle \mathcal{A}^*(u), X\rangle_F = \langle u, \mathcal{A}(X)\rangle,\ \forall X \in \mathbb{R}^{n_r\times n_c}, u \in \mathbb{R}^m.$$

Define $\mathcal{B} : \mathbb{R}^{n_r\times n_c} \to \mathbb{R}^{n_r\times n_c}$ by $\mathcal{B} = \mathcal{A}^*\mathcal{A} - I$, and the constant μ by

$$\mu = \sup_{Z\in\mathcal{M}(k)} \|\mathcal{B}(Z)\|_F.$$

If $\epsilon \leq 1$, then $1/\epsilon \geq 1$, and $1 + 6/\epsilon \leq 7/\epsilon$. Accordingly, define

$$d(\epsilon) = \left(\frac{7}{\epsilon}\right)^{(n_r+n_c+1)k} \geq \left(1 + \frac{6}{\epsilon}\right)^{(n_r+n_c+1)k}.$$

Let $\mathcal{S}_\epsilon(k)$ denote a minimal ϵ-cover of $\mathcal{S}(k)$ with respect to $\|\cdot\|_F$, and note that $|\mathcal{S}_\epsilon(k)| \leq d(\epsilon)$ by Theorem 7.20. Therefore, by the union of events bound, we get

$$\Pr\left\{\max_{Y\in\mathcal{S}_\epsilon(k)} |\|\mathcal{A}(Y)\|_2^2 - 1| > t\right\} \leq 2|\mathcal{S}_\epsilon(k)|\exp(-\tilde{c}mt^2) \leq 2d(\epsilon)\exp(-\tilde{c}mt^2).$$

Now let $X \in \mathcal{S}(k)$ be arbitrary, and choose $Y \in \mathcal{S}_\epsilon(k)$ such that $\|X - Y\|_F \leq \epsilon$. Observe that

$$\langle X, \mathcal{B}(X)\rangle_F = \langle X, (\mathcal{A}^*\mathcal{A} - I)(X)\rangle_F = \|\mathcal{A}(X)\|_2^2 - 1.$$

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Next, note that

$$\langle X, \mathcal{B}(X)\rangle_F = \langle Y, \mathcal{B}(Y)\rangle_F + \langle X - Y, \mathcal{B}(X+Y)\rangle_F. \tag{7.57}$$

Therefore

$$\begin{aligned} |\langle X - Y, \mathcal{B}(X+Y)\rangle_F| &\leq \|X - Y\|_F \cdot \|\mathcal{B}(X-Y)\|_F \\ &\leq \|X - Y\|_F \cdot (\|\mathcal{B}(X)\|_F + \|\mathcal{B}(Y)\|_F) \\ &\leq 2\mu\epsilon, \end{aligned}$$

while

$$|\langle Y, \mathcal{B}(Y)\rangle_F| = |\|\mathcal{A}(Y)\|_2^2 - 1| \leq t.$$

Substituting these bounds into (7.57) shows that

$$|\langle X, \mathcal{B}(X)\rangle_F| \leq t + 2\mu\epsilon, \ \forall X \in \mathcal{S}(k). \tag{7.58}$$

Now take the supremum of the left side with respect to X, and note that

$$\sup_{X \in \mathcal{S}(k)} \langle X, \mathcal{B}(X)\rangle_F| = \mu$$

because $\mathcal{B}$ is self-adjoint. Therefore (7.58) implies that

$$\mu \leq t + 2\epsilon\mu, \text{ or } \mu \leq \frac{t}{1-2\epsilon}.$$

Hence

$$|\langle X, \mathcal{B}(X)\rangle_F| \leq t + 2\mu\epsilon \leq t + \frac{2t\epsilon}{1-2\epsilon} = \frac{t}{1-2\epsilon}, \ \forall X \in \mathcal{S}(k).$$

So we can conclude that

$$\Pr\left\{\max_{X \in \mathcal{S}_\epsilon(k)} |\|\mathcal{A}(X)\|_2^2 - 1| > t/(1-2\epsilon)\right\} \leq 2d(\epsilon)\exp(-\tilde{c}mt^2).$$

Therefore, given an RRIP constant δ and failure probability ξ, we can take $t = (1-2\epsilon)\delta$ where ϵ is as yet unspecified. Then

$$\Pr\left\{\max_{Y \in \mathcal{S}_\epsilon(k)} |\|\mathcal{A}(Y)\|_2^2 - 1| > \delta\right\} \leq 2d(\epsilon)\exp(-\tilde{c}m(1-2\epsilon)^2\delta^2). \tag{7.59}$$

Let us now substitute for $d(\epsilon)$. Then the right side is $\leq \xi$ provided

$$2\left(\frac{7}{\epsilon}\right)^{(n_r+n_c+1)k} \exp(-\tilde{c}m(1-2\epsilon)^2\delta^2) \leq \xi,$$

or

$$m \geq \frac{1}{\tilde{c}(1-2\epsilon)^2\delta^2}\left[(n_r+n_c+1)k\ln\frac{7}{\epsilon} + \ln\frac{2}{\xi}\right]. \tag{7.60}$$

We are free to choose *any* $\epsilon \in (0, 0.5)$ in the above. To get a good bound, let us minimize

$$f(\epsilon) = \frac{1}{(1-2\epsilon)^2}\ln\frac{7}{\epsilon},$$

which is the only ϵ-dependent part of the bound, with respect to ϵ. Figure 7.1 shows a plot of $f(\epsilon)$ versus ϵ. From this plot it can be seen that if we take $\epsilon = 0.05$, then $f(\epsilon) \leq 7$. Moreover

$$\frac{1}{(1-2\epsilon)^2} \leq 1.25 = \frac{5}{4}.$$

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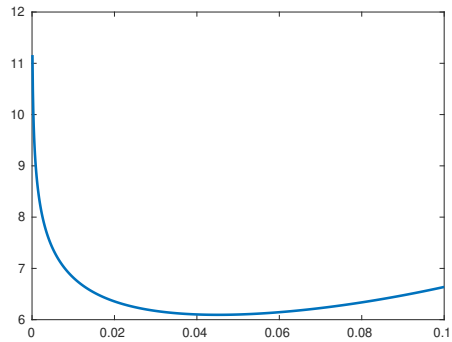

Figure 7.1. *Plot of $f(\epsilon)$ versus ϵ.*

Therefore the bound in (7.60) becomes

$$m \geq \frac{1}{\tilde{c}\delta^2}\left[7k(n_r + n_c + 1) + \frac{5}{4}\ln\frac{2}{\xi}\right],$$

which is the same as (7.56).

Now we present an improvement of Theorem 7.21 for the case where the underlying random variable is a normal Gaussian instead of an arbitrary sub-Gaussian variable. The main difference is that in this case, when $\|X\|_F = 1$, the quantity $m\|\mathcal{A}(X)\|_2^2$ follows a chi-squared distribution with m degrees of freedom. Therefore it follows from [153, Lemma 1] that (7.54) can be replaced by

$$\Pr\{|\|\mathcal{A}(X)\|_2^2 - 1| > t\} \leq 2\exp(-c(t)m), \tag{7.61}$$

where[28]

$$c(t) = \frac{t^2}{4} - \frac{t^3}{6}. \tag{7.62}$$

By using this improved estimate of the tail probability, we can give an alternate bound for the sample complexity.

Theorem 7.22. *Define $\mathcal{A} : \mathbb{R}^{n_r \times n_c} \to \mathbb{R}^m$ as in Theorem 7.19, where the underlying random variable is normal Gaussian. If*

$$m \geq \frac{12}{\delta^2}\left[7k(n_r + n_c + 1) + \frac{5}{4}\ln\frac{2}{\xi}\right], \tag{7.63}$$

then $\mathcal{A}$ of (7.51) *satisfies the RRIP of order k with constant δ, with probability $\geq 1 - \xi$.*

The proof is entirely parallel to that of Theorem 7.21 until (7.59). At this point, with $t = (1 - 2\epsilon)\delta$, the exponent on the right side changes to

$$\tilde{c}m(1 - 2\epsilon)^2\delta^2 \leftarrow m\left[\frac{(1 - 2\epsilon)^2\delta^2}{4} - \frac{(1 - 2\epsilon)^3\delta^3}{6}\right],$$

where (A) $\leftarrow$ (B) means that (B) replaces(A). Consequently, in (7.60),

$$\frac{1}{\tilde{c}(1 - 2\epsilon)^2\delta^2} \leftarrow \left[\frac{(1 - 2\epsilon)^2\delta^2}{4} - \frac{(1 - 2\epsilon)^3\delta^3}{6}\right]^{-1}.$$

[28]Not that there is a typo in the line after [56, Equation (II.3)].

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Now the expression inside the brackets can be written as

$$\frac{(1-2\epsilon)^2\delta^2}{4}\left(1-\frac{2(1-2\epsilon)\delta}{3}\right).$$

Next, for $\epsilon, \delta \in (0,1)$, we have that

$$1-\frac{2(1-2\epsilon)\delta}{3} \geq 1-\frac{2\delta}{3} \geq \frac{1}{3}.$$

Therefore

$$\frac{(1-2\epsilon)^2\delta^2}{4}\left(1-\frac{2(1-2\epsilon)\delta}{3}\right) \geq \frac{(1-2\epsilon)^2\delta^2}{12}.$$

So the sample complexity bound (7.60) now becomes

$$m \geq \frac{12}{\delta^2}\left[7k(n_r+n_c+1)+\frac{5}{4}\ln\frac{2}{\xi}\right],$$

which is the same as (7.60) with $\tilde{c}$ replaced by $1/12$. Hence (7.56) gets replaced by (7.63).

Remarks:

1. Note that the estimate (7.63) is rather crude because we wish to permit *any* $\delta \in (0,1)$. However, large values of δ in the RRIP are not realistic. If we were to restrict δ to belong to $(0, 0.5)$, for example, then in the above proof

$$1-\frac{2\delta}{3} \geq \frac{2}{3},$$

which would allow us to replace the factor 12 in (7.63) by 6.

2. By Theorem 6.16, a normal Gaussian variable satisfies (7.52) with $c = 1/2$. If we were to compute the corresponding constant $\tilde{c}$ using (7.53), we would get $\tilde{c} = 0.0201$, and $1/\tilde{c} \approx 50$. Hence, if we insist on a bound that holds for *all* $\delta \in (0,1)$, the bound in (7.63) improves that in (7.56) by roughly a factor of 4. If we are content with a bound that holds for $\delta \in (0, 0.5)$, then the bound (7.63) with 12 replaced by 6 improves that in (7.56) by roughly a factor of 8.

We conclude this section by briefly describing (without proofs) another approach to matrix recovery.

Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is the unknown matrix that we wish to recover using probabilistic methods. In (7.51), the data consists of the Frobenius inner products $\langle A_1, X\rangle_F, \ldots, \langle A_m, X\rangle_F$, where each A_i is a random matrix in $\mathbb{R}^{n_r \times n_c}$. In such a case, the computation of the inner product $\langle A_i, X\rangle_F$ can be time-consuming. In [50], it is suggested to choose each A_i to be a *rank-one matrix* of the form $A_i = b_i c_i^\top$. In this case it is easy to show that

$$\langle b_i c_i^\top, X\rangle_F = b_i^\top X c_i,$$

which is just a standard triple product. The question studied in [50] is whether such rank-one projections suffice to recover an unknown matrix X. The answer is shown to be in the affirmative. For this purpose, the RIP is replaced by the robust uniform boundedness (RUB) property. Complete details can be found in the paper.

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7.4 ▪ Matrix Completion: Probabilistic Methods

In this section and the next, the emphasis is on the so-called matrix *completion* problem, as opposed to the matrix *recovery* problem that has been the object of study until now. To state the matrix completion problem precisely, we begin with some notation. Suppose the unknown matrix X to be recovered belongs to $\mathbb{R}^{n_r \times n_c}$, and assume without loss of generality that $n_r \leq n_c$. If $n_r > n_c$, X can be replaced by its transpose. For each index pair $(i,j) \in [n_r] \times [n_c]$, define $E_{i,j}$ to be the binary matrix with a 1 in position (i,j) and zeros elsewhere. Then the collection of matrices $\{E_{i,j}, (i,j) \in [n_r] \times [n_c]\}$ defines an orthonormal basis for $\mathbb{R}^{n_r \times n_c}$ in the Frobenius inner product. Moreover, $\langle E_{i,j}, X \rangle_F = X_{i,j}$ for all $X \in \mathbb{R}^{n_r \times n_c}$. Now suppose a set $\Omega \subseteq [n_r] \times [n_c]$, called the **measurement set**, is specified. To be specific suppose that $|\Omega| = m$ and that $\Omega = \{(i_1, j_1), \ldots, (i_m, j_m)\}$. In the matrix completion problem, the m measurements consist of $\langle E_{i,j}, X \rangle_F = X_{i,j}$ for all $(i,j) \in \Omega$, or, equivalently, the values $X_{i,j}$ for all $(i,j) \in \Omega$. The set of measurements can be equivalently expressed as $E_\Omega.X$ where

$$E_\Omega = \sum_{(i,j) \in \Omega} E_{i,j}$$

has an element of 1 if $(i,j) \in \Omega$ and 0 otherwise, and $A.B$ denotes the **Hadamard product**. Recall that if A, B have the same dimensions, then $C = A.B$ is defined by $c_{ij} = a_{ij} b_{ij}$ for all (i,j). With these conventions, the **matrix completion problem** can be stated as follows:

$$\hat{X} = \arg\min \operatorname{rank}(Z) \text{ s.t. } E_\Omega.Z = E_\Omega.X. \tag{7.64}$$

The above problem is a special case of minimizing the rank of an unknown matrix subject to linear constraints. The general problem is NP-hard, and specializing the constraints to the above form does not make it any easier—it is still NP-hard. Therefore we replace the rank function by its convex relaxation, which is the nuclear norm, and replace the problem in (7.64) by

$$\hat{X} = \arg\min \|Z\|_N \text{ s.t. } E_\Omega.Z = E_\Omega.X. \tag{7.65}$$

The issue, as always, is to find conditions under which the solution to the problem in (7.65) is the same as that of (7.64). It turns out that there are some new wrinkles in the matrix completion problem that are not present in the general problem of matrix recovery.

7.4.1 ▪ The Coherence of a Matrix

One such key concept is called "coherence," introduced in [54]. To lead up to this concept, suppose the unknown matrix X has an entry of 1 at just one location, with the rest of the elements being equal to zero. Without loss of generality, suppose that $X_{1,1} = 1$ while $X_{i,j} = 0$ for $(i,j) \neq (1,1)$. In this case X has rank one. Moreover, if one were to sample X by computing an inner product $\langle B, X \rangle_F$ for some random Gaussian matrix B, then with probability one it can be stated that $B_{1,1} \neq 0$, so that $\langle B, X \rangle_F \neq 0$. On the other hand, if we were to study the matrix *completion* problem by sampling various elements of X, then all but one of the sampled elements would equal zero. Consequently, unless $(1,1)$ belongs to the sample set Ω, the solution $\hat{X}$ to (7.64) (or (7.65) for that matter) would be the zero matrix, which would not equal X.

The source of difficulty in this case is that the matrix X has very high coherence. In [54], two distinct measures of coherence are defined.

Definition 7.23. *Suppose $X \in \mathbb{R}^{n_r \times n_c}$ has rank r and the reduced singular value decomposition $X = U \Sigma V^\top$, where $U \in \mathbb{R}^{n_r \times r}$, $V \in \mathbb{R}^{n_c \times r}$, and $\Sigma \in \mathbb{R}^{r \times r}$ is the diagonal matrix of singular values of X. Let $\mathcal{P}_U = UU^\top \in \mathbb{R}^{n_r \times n_r}$ denote the orthogonal projection of $\mathbb{R}^{n_r}$ onto $U\mathbb{R}^{n_r}$.*

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Finally, let $\mathbf{e}_i \in \mathbb{R}^{n_r}$ *denote the* i*th canonical basis vector. Then we define*

$$\mu_0(U) := \frac{n_r}{r} \max_{i \in [n_r]} \|\mathcal{P}_U \mathbf{e}_i\|_2^2. \tag{7.66}$$

The quantity $\mu_0(V)$ *is defined analogously, and*

$$\mu_0(X) := \max\{\mu_0(U), \mu_0(V)\}. \tag{7.67}$$

Next, we define

$$\mu_1(X) := \sqrt{\frac{n_r n_c}{r}} \|UV^\top\|_\infty, \tag{7.68}$$

where

$$\|M\|_\infty := \max_{i \in [n_r], j \in [n_c]} |M_{i,j}|.$$

The coherence $\mu_0(U)$ measures how closely any one of the n_r canonical basis vectors is aligned with the columns of U. Note that an equivalent characterization of $\mu_0(U)$ is the following: Let $U^i, i \in [n_r]$, denote the ith row of the matrix U. Then

$$\mu_0(U) = \frac{n_r}{r} \max_{i \in [n_r]} \|U^i\|_2^2.$$

It is easy to see that if one of the columns of U equals one of the $\mathbf{e}_i$, then $\|\mathcal{P}_U \mathbf{e}_i\|_2^2 = 1$ and $\mu_0(U) = n_r/r$. Indeed n_r/r is the maximum value for $\mu_0(U)$. This is the difficulty with the matrix whose $(1,1)$ entry is 1 and the rest are zero. The SVD of this rank-one matrix has $U = \mathbf{e}_1$, and, as a result, this matrix has maximum coherence. On the other side, the minimum value for $\mu_0(U)$ is 1 and is achieved when every element of U has equal magnitude $1/\sqrt{n_r}$. One example of a matrix with minimum coherence is provided by the class of **Hadamard matrices**. Given two matrices A, B of whatever dimensions, it is possible to define their **Kronecker product** as follows: Suppose, to be specific, that A has dimensions $k \times l$ and B has dimensions $r \times s$. Then the Kronecker product $A \otimes B$ has dimensions $kr \times ls$ and is defined as

$$\begin{bmatrix} a_{11}B & \dots & a_{1l}B \\ \vdots & \vdots & \vdots \\ a_{k1}B & \dots & a_{kl}B \end{bmatrix}.$$

The Hadamard matrix $H(l)$ has dimensions $2^l \times 2^l$ for every integer $l \geq 1$ and is defined recursively as follows:

$$H(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H(l) = H(1) \otimes H(l-1).$$

The elements of $H(l)$ are all equal to ± 1, and the columns are orthogonal to each other. Therefore it readily follows that the normalized matrix $(1/2^{l/2})H(l)$ is orthogonal and has minimum coherence of 1. Moreover, every submatrix consisting of all rows and some subset of columns also has minimum coherence. As for $\mu_1(X)$, it is easy to derive a quick (and possibly not very tight) bound. The (ij)th element of $UV^\top$ can be bounded via Schwarz's inequality as

$$\begin{aligned} |(UV^\top)_{ij}| &= \left| \sum_{k \in [r]} u_{ik} v_{jk} \right| = |\langle V^j, U^i \rangle| \leq \|V^j\|_2 \cdot \|U^i\|_2 \\ &\leq \sqrt{\frac{\mu_0(U)\mu_0(V)}{n_r n_c}} r \leq \frac{\mu_0(X) r}{\sqrt{n_r n_c}}. \end{aligned}$$

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Note that all of the preceding discussion applies irrespective of whether the sample set Ω is chosen in a deterministic fashion or in a random fashion. As the title of this section indicates, the emphasis in the present section is on matrix completion when the elements of the sampling set Ω are chosen at random. This is in contrast with methods where the sampling set Ω is chosen in a deterministic fashion, namely corresponding to the edges in a Ramanujan graph; that approach is studied in the next section. Within probabilistic methods, it is possible to make a further distinction. Some authors replace the rank function in (7.64) by the nuclear norm because (as shown in Theorem 1.53) the convex relaxation of the rank function over the unit sphere in the spectral norm is indeed the nuclear norm. There is, however, another method called "OptSpace" that does not use nuclear norm minimization. Both methods are discussed here.

7.4.2 ▪ Optimality Conditions for Nuclear Norm Minimization

In this subsection we first derive a necessary and sufficient condition for a matrix X to be a solution of the constrained minimization problem in (7.65). However, this condition is not very easy to apply. Therefore we present a sufficient condition that guarantees that X is a *unique* solution to (7.65).

Theorem 7.24. *Suppose X has rank r and let $U\Sigma V^\top$ be a reduced SVD of X. Choose matrices such that $[U \;\; U_\perp]$, $[V \;\; V_\perp]$ are both square orthogonal matrices. Then X is a solution of the constrained minimization problem in* (7.65) *if and only if there exists a matrix $M \in \mathbb{R}^{(n_r-r)\times(n_c-r)}$ with $\|M\|_S \leq 1$, such that if we define $Y = UV^\top + U_\perp M V_\perp^\top$, then we have*

$$Y_{ij} = 0, \; \forall (i,j) \notin \Omega. \tag{7.69}$$

Proof. Let us convert the constrained minimization problem in (7.65) to Lagrangian form by defining

$$J = \|Z\|_N + \sum_{(i,j)\in\Omega} \lambda_{ij}(\langle E_{ij}, Z\rangle_F - \langle E_{ij}, X\rangle_F),$$

where E_{ij} denotes the matrix with a 1 in position (i,j) and zeros elsewhere. Because this Lagrangian is convex, a matrix X is a solution of (7.65) if and only if $0 \in \partial J(X)$. Thus it is just a matter of computing the subdifferential of J at X and checking whether the zero matrix belongs to the subdifferential.

Now

$$\partial J(X) = \partial \|\cdot\|_N(X) + \sum_{(i,j)\in\Omega} \lambda_{ij} E_{ij}.$$

We can invoke Corollary 1.65 to see that

$$\partial \|\cdot\|_N(X) = \{UV^\top + U_\perp M V_\perp^\top, \|M\|_S \leq 1\}.$$

Therefore

$$\partial J(X) = \{UV^\top + U_\perp M V_\perp^\top + W, \|M\|_S \leq 1\},$$

where

$$W = \sum_{(i,j)\in\Omega} \lambda_{ij} E_{ij}.$$

Now X is a solution of (7.65) if and only if the zero matrix belongs to this set. Let us define $Y = UV^\top + U_\perp M V_\perp^\top$, and note that $W_{ij} = 0$ whenever $(i,j) \notin \Omega$. Thus, in order for $Y + W$ to equal zero, we must have that $Y_{ij} = 0$ whenever $(i,j) \notin \Omega$. The values of $Y_{ij}, (i,j) \in \Omega$ do not matter because we can choose $\lambda_{ij} = -Y_{ij}$.

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While the condition in Theorem 7.24 is both necessary as well as sufficient, it is not that easy to use. Moreover, it guarantees only that X is a solution of (7.65)—not that X is a *unique* solution. Now we present a sufficient condition under which the true matrix X is the unique solution to the nuclear norm minimization problem (7.65). Such sufficient conditions are to be found in various papers on matrix completion. The next theorem is roughly the same as [207, Theorem 2] and is stated in this form in [41, Lemma 2]. Note that the theorem is "deterministic" and there is nothing "random" about the theorem.

To state the theorem, suppose $X = U\Sigma V^\top$ is the unknown matrix of rank r or less that is to be recovered, where $U \in \mathbb{R}^{n_r \times r}$, $V \in \mathbb{R}^{n_c \times r}$, and Σ is diagonal of dimensions $r \times r$. Define $\mathcal{T} \subseteq \mathbb{R}^{n_r \times n_c}$ to be the subspace spanned by all matrices of the form $UB^\top$ and $CV^\top$. It is easy to show that the projection operator $\mathcal{P}_\mathcal{T}$ equals

$$\begin{aligned}\mathcal{P}_\mathcal{T} Z &= UU^\top Z + ZVV^\top - UU^\top ZVV^\top \\ &= UU^\top Z + U_\perp U_\perp^\top ZVV^\top \\ &= UU^\top ZV_\perp V_\perp^\top + ZVV^\top,\end{aligned}$$

where $U_\perp U_\perp^\top = I_{n_r} - UU^\top$ and $V_\perp V_\perp^\top = I_{n_c} - VV^\top$.

Theorem 7.25. *Define $\alpha = r/n_r$. Suppose there exists a $Y \in \mathbb{R}^{n_r \times n_c}$ such that the following hold:*

1. *Y belongs to the image of $E_\Omega.$, that is $Y_{ij} = 0,\ \forall (i,j) \notin \Omega$.*
2. *Y satisfies*

$$\|\mathcal{P}_\mathcal{T} Y - UV^\top\|_F \le \sqrt{\frac{\alpha}{32}}, \quad \|\mathcal{P}_{\mathcal{T}^\perp}(Y)\|_S < \frac{3}{4}. \tag{7.70}$$

Suppose further that the operator norm of $(1/\alpha)\mathcal{P}_\mathcal{T} E_\Omega. - I$ when restricted to the subspace $\mathcal{T}$ is no larger than $1/2$. In other words

$$\|(1/\alpha)\mathcal{P}_\mathcal{T} E_\Omega.Z - Z\|_F \le (1/2)\|Z\|_F,\ \forall Z \in \mathcal{T}. \tag{7.71}$$

Under these assumptions, for any $\Delta \in \mathbb{R}^{n_r \times n_c} \setminus \{0\}$ such that $E_\Omega.\Delta = 0$, we have that

$$\|X + \Delta\|_N > \|X\|_N, \tag{7.72}$$

so that $\hat{X} = X$ is the unique solution to (7.65).

Proof. Suppose $E_\Omega.\Delta = 0$, so that $\|E_\Omega.\Delta\|_F = 0$. Then

$$\begin{aligned}\|E_\Omega.\mathcal{P}_\mathcal{T}\Delta\|_F^2 &= \langle E_\Omega.\mathcal{P}_\mathcal{T}\Delta, \mathcal{P}_\mathcal{T}\Delta\rangle_F \\ &= \langle \mathcal{P}_\mathcal{T} E_\Omega.\mathcal{P}_\mathcal{T}\Delta - \alpha\mathcal{P}_\mathcal{T}\Delta, \mathcal{P}_\mathcal{T}\Delta\rangle_F + \alpha\langle \mathcal{P}_\mathcal{T}\Delta, \mathcal{P}_\mathcal{T}\Delta\rangle_F \\ &\ge^{(a)} \alpha\|\mathcal{P}_\mathcal{T}\Delta\|_F^2 - \alpha/2\|\mathcal{P}_\mathcal{T}\Delta\|_F^2 = \alpha/2\|\mathcal{P}_\mathcal{T}\Delta\|_F^2,\end{aligned}$$

where (a) follows from (7.71). Now, since $\|E_\Omega.\Delta\|_F = 0$, we have $\|E_\Omega.\mathcal{P}_\mathcal{T}\Delta\|_F = \|E_\Omega.\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_F$. Therefore,

$$\begin{aligned}\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N &\ge \|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_F \ge \|E_\Omega.\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_F \\ &\ge \sqrt{\alpha/2}\|\mathcal{P}_\mathcal{T}\Delta\|_F.\end{aligned} \tag{7.73}$$

Note that (7.73) implies that $\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N > 0$. Suppose that $\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N = 0$. Then (7.73) implies that $\|\mathcal{P}_\mathcal{T}\Delta\|_F = 0$, and in turn $\Delta = \mathcal{P}_{\mathcal{T}^\perp}\Delta + \mathcal{P}_\mathcal{T}\Delta = 0$, which is a contradiction.

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Next, recall that for any matrix M, it is true that

$$\|M\|_N = \max_{U',V'} \langle U'V'^\top, M\rangle_F$$

over all matrices U', V' with orthogonal columns. In particular, for a specific Δ, it is possible to choose $U_\perp, V_\perp$ such that $[U \;\; U_\perp]$, $[V \;\; V_\perp]$ have orthogonal columns, and

$$\langle U_\perp V_\perp^\top, \mathcal{P}_{\mathcal{T}^\perp}\Delta\rangle_F = \|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N.$$

For such a choice, we have

$$\begin{aligned}
\|X+\Delta\|_N &\geq^{(a)} \langle UV^\top + U_\perp V_\perp^\top, X+\Delta\rangle_F \\
&=^{(b)} \|X\|_N + \langle UV^\top + U_\perp V_\perp^\top, \Delta\rangle_F \\
&=^{(c)} \|X\|_N + \langle UV^\top + U_\perp V_\perp^\top, \Delta\rangle_F - \langle Y, \Delta\rangle_F \\
&= \|X\|_N + \langle UV^\top - \mathcal{P}_{\mathcal{T}}Y, \mathcal{P}_{\mathcal{T}}\Delta\rangle_F + \langle U_\perp V_\perp^\top - \mathcal{P}_{\mathcal{T}^\perp}Y, \mathcal{P}_{\mathcal{T}^\perp}\Delta\rangle_F \\
&\geq^{(d)} \|X\|_N - \|UV^\top - \mathcal{P}_{\mathcal{T}}Y\|_F\|\mathcal{P}_{\mathcal{T}}\Delta\|_F + \|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N - \|\mathcal{P}_{\mathcal{T}^\perp}Y\|_S\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N \\
&\geq \|X\|_N - \sqrt{\alpha/32}\|\mathcal{P}_{\mathcal{T}}\Delta\|_F + (1-\|\mathcal{P}_{\mathcal{T}^\perp}Y\|_S)\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N, \qquad (7.74)
\end{aligned}$$

where (a) follows from the characterization of the nuclear norm, (b) follows from $\langle U_\perp V_\perp^\top, X\rangle_F = 0$, (c) follows from $\langle Y, \Delta\rangle_F = 0$, and (d) follows from Hölder's inequality. Now it follows from (7.70) and (7.73) that

$$(1-\|\mathcal{P}_{\mathcal{T}^\perp}Y\|_S)\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N > (1/4)\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N \geq \sqrt{\alpha/32}\|\mathcal{P}_{\mathcal{T}}\Delta\|_F,$$

where we use the fact that $\|\mathcal{P}_{\mathcal{T}^\perp}\Delta\|_N > 0$. Substituting this fact into the last equation in (7.74) shows that $\|X+\Delta\|_N > \|X\|_N$.

7.4.3 ▪ Matrix Completion via Nuclear Norm Minimization

Given that the optimization problem in (7.64) is NP-complete, a logical approach is to replace the rank function by its convex relaxation (7.65). Specifically, we are interested in the question: Under what conditions is the true but unknown matrix X the unique solution to the optimization problem in (7.65)? In this setting one can distinguish between two distinct situations. If one were to sample m out of the $n_r n_c$ elements of the unknown matrix X *without* replacement, then one is guaranteed that exactly m distinct elements of X are measured. However, the disadvantage is that the locations of the m samples are not independent, because once the first element has been selected, there are only $n_r n_c - 1$ choices for the second sample, and so on. Thus sampling without replacement requires quite advanced probabilistic analysis. This is the approach adopted in [54]. An alternative is to sample the elements of X *with* replacement. In this case the locations of the m samples are indeed independent. However, the price to be paid is that, with some small probability, there would be duplicate samples, so that after m random draws, the number of elements of X that are measured could be smaller than m. This is the approach adopted in [207]. On balance, the approach of sampling with replacement is easier to analyze. Therefore, in the present section, *we present without proof* the main results of [54] and [207], while referring the reader to the original papers for full details. Note that, when the sampling matrix is generated at random, the recovery of the unknown matrix can be guaranteed only *with high probability* that can be made close to, but not exactly equal to, one.

Theorem 7.26. *(See [54, Theorem 1.3].) Suppose $X \in \mathbb{R}^{n_r \times n_c}$ has rank r, and let $n = \max\{n_r, n_c\}$. Suppose that m elements of X are sampled (without replacement), and define $\hat{X}$*

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as in (7.65). *There exist constants* C, c *such that, if*

$$m \geq C \max\{\mu_1^2, \mu_0^{1/2}\mu_1, \mu_0 n^{1/4}\} nr\beta \log n \tag{7.75}$$

for some $\beta > 2$, *then* $\hat{X} = X$ *is the unique minimizer of the optimization problem in* (7.65), *with probability at least equal to* $1 - cn^{-\beta}$. *For* $r \leq \mu_0^{-1} n^{1/5}$ *this estimate can be sharpened to*

$$m \geq C\mu_0 n^{6/5} r\beta \log n \tag{7.76}$$

with the same probability of success.

Now let us discuss the implications of this theorem. A general matrix of dimensions $n_r \times n_c$ has $n_r n_c$ free parameters. However, because a matrix X of rank r can be factorized as $X_l X_r$ where $X_l \in \mathbb{R}^{n_r \times r}$ and $X_r \in \mathbb{R}^{r \times n_c}$, a rank r matrix has no more than $r(n_r + n_c)$ free parameters.[29] If $n_r = n_c$ so that the matrix is square, the number of free parameters in a general matrix is n^2 whereas the number of free parameters in a rank r matrix is no larger than $2rn$. Thus the bounds for the number of measurements m can be compared against both of these numbers of free parameters.

By taking just the last term inside the maximum operator in (7.75), we get the lower bound

$$m \geq C\mu_0 n^{5/4} r\beta \log n.$$

Given that m would have to grow at least linearly with respect to the number of free parameters (i.e., $m = \Omega(nr)$), the extra factor here is $n^{1/4} \log n$. In the other direction, if $r = O(n^{1/2})$, for example, then $m = O(n^{7/4})$, which is not substantially smaller than n^2. On the other hand, if r is kept fixed while n is increased, then m can be quite a bit smaller than n^2. The improved bound in (7.76), where $n^{5/4}$ is replaced by $n^{6/5}$, requires that $r \leq n^{1/5}$ (because $\mu_0 \geq 1$). Thus, in order to be useful, the unknown matrix has to have very small rank. To illustrate, suppose $n = 2^{20} \approx 10^6$. Then $r \leq 2^4 = 16$. If we take a more realistic size of $n = 2^{10} = 1,024$, then $r \leq 2^2 = 4$. Therefore the bounds on the ranks under which Theorem 7.26 is *guaranteed* to achieve (probabilistic) recovery are unrealistically low.

Now we present the main result from [207].

Theorem 7.27. *(See [207, Theorem 2].) Choose some constant* $\beta > 1$, *and draw*

$$m \geq 32 \max\{\mu_1^2, \mu_0\} r(n_r + n_c)\beta \log^2(2n_c) \tag{7.77}$$

samples from $[n_r] \times [n_c]$ *with replacement. Define* $\hat{X}$ *as in* (7.65). *Then, with probability at least equal to* $1 - \zeta$ *where*

$$\zeta = 6\log(n_c)(n_r + n_c)^{2-2\beta} + n_c^{2-2\sqrt{\beta}}, \tag{7.78}$$

the optimization problem has a unique solution $\hat{X}$, *and* $\hat{X} = X$.

Comparing the bounds in (7.75) and (7.78) with $n_r = n_c = n$, we see that the term $n^{5/4} r \log n$ in the former is replaced by $nr \log^2 n_c$ after ignoring various constants. Also, unlike in (7.75), the constants in (7.78) are explicitly displayed. Thus, in addition to the proof being more straightforward, the bound is also better in the case of Theorem 7.27.

The key point to note is that the conditions in Theorems 7.26 and 7.27 are only *sufficient* conditions. As with the basis pursuit formulation, it is possible to carry out some simulations on phase transitions for nuclear norm minimization. This is done in [41], and the simulations show that these conditions are in reality quite conservative, especially when the locations to be sampled are selected in a deterministic fashion.

[29] Actually the number is smaller, but this estimate is good enough for present purposes.

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7.4.4 ▪ The OptSpace Algorithm

In this subsection we describe the OptSpace algorithm from [147, 148], a new approach to matrix completion that *does not depend* on minimizing the nuclear norm. While the algorithm is easy to describe, its analysis is quite advanced. Specifically, the algorithm involves optimizing a nonconvex function on a compact manifold using a steepest descent technique, and thus involves a great many technical details. For this reason, we describe the algorithm and refer the reader to the original publications for the analysis.

There are two papers that describe the OptSpace algorithm, namely [147] and [148]. In [147], it is assumed that the unknown matrix to be completed is "truly" row rank, and not "nearly" low rank, and also that the measurements are noise-free. Both of these assumptions are removed in [148]. In the interest of simplicity, we describe only the simpler version of OptSpace as given in [147].

As before, let $X \in \mathbb{R}^{n_r \times n_c}$ be the unknown matrix to be recovered, and let r be a known upper bound on its rank. Let $\Omega \subseteq [n_r] \times [n_c]$ be a measurement set, and let E_Ω denote the corresponding measurement matrix. Let $G = E_\Omega.X$ denote the set of measurements available to the learner. The OptSpace algorithm consists of three steps.

Trimming: The first step is to remove entries from columns (or rows) that are over-represented in the random sampling. If $|\Omega|$ is the total number of samples, then $|\Omega|/n_r$ is the average number of samples per column. If any column contains more than twice this many samples, then all entries in the column are set equal to zero, and the resulting "trimmed" matrix is denoted by $\tilde{G}$.

Projection: Construct the best rank-r projection of the trimmed matrix $\tilde{G}$, say $\tilde{G} = U_0 S_0 V_0^\top$, using a singular value decomposition.

Cleaning residual errors: Once the trimmed matrix is projected onto the set of rank r matrices, the rank-r matrix $U_0 S_0 V_0^\top$ is not necessarily a good approximation to the *original* measurement matrix G. This is addressed by solving an optimization problem. Define

$$\mathcal{F}(U, S, V) = \|E_\Omega.(G - USV^\top)\|_F^2,$$

and define the function of U and V alone as

$$F(U, V) = \min_{S \in \mathbb{R}^{r \times r}} \mathcal{F}(U, S, V).$$

Note that, for fixed U, V, the function $\mathcal{F}(U, S, V)$ is quadratic in S. Hence it is straightforward to compute $F(U, V)$. However, $F(U, V)$ is a highly nonlinear function in its arguments. In the cleaning step, the aim is to minimize $F(U, V)$ over all matrices U, V with orthonormal columns, that is, the set of all U, V that satisfy $U^\top U = V^\top V = I_r$. It is suggested in [147] to do this using steepest descent, starting at (U_0, V_0).

In [148], the matrix X is not assumed to be exactly of rank r, and it is permitted to have measurement noise, so that the information available to the learner consists of $E_\Omega.X + W$, where W is a noise matrix. The OptSpace algorithm proceeds along similar lines, with just a few extra modifications to cater to the more general situation.

7.5 ▪ Matrix Completion: Deterministic Methods

In the previous section the focus was on matrix completion when the elements to be measured were chosen at random. In the present section, we study the case where the measurement matrix is chosen in a deterministic fashion, specifically, as the biadjacency matrix of a Ramanujan graph. Two types of results are presented. First, it is shown that relatively easily constructed matrices provide a decent approximation to the unknown matrix. Second, it is shown that, under suitable

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conditions, it is possible to recover the unknown matrix exactly.[30] In either case, Ramanujan bigraphs play a central role, so we begin with a study of this topic.

7.5.1 ▪ Ramanujan Bigraphs

Suppose $B \in \{0,1\}^{n_r \times n_c}$. Then B can be interpreted as the biadjacency matrix of a bipartite graph with n_r vertices on one side and n_c vertices on the other side. By convention, the side with more vertices is called the "left" side and the other is the "right" side. In what follows, it is assumed that $n_r \leq n_c$, so that columns of B correspond to the left side and rows to the right side. The bipartite graph associated with B (that is, the bipartite graph for which B is the biadjacency matrix) is said to be **left-regular** if every left vertex has the same degree, **right-regular** if every right vertex has the same degree, and **biregular** if it is both left-regular and right-regular. Due to the convention that $n_r \leq n_c$, left-regularity is equivalent to the requirement that every column of B has the same number of ones, and right-regularity is equivalent to the requirement that every row has the same number of ones. By a slight abuse of language, we associate left- and right-regularity with the matrix B itself, as well as the bipartite graph corresponding to B.

Suppose $B \in \{0,1\}^{n_r \times n_c}$ is biregular with right (or row) degree d_r and left (or column) degree d_c. We refer to such graphs as (d_r, d_c)-biregular. It is clear that $n_r d_r = n_c d_c$, or equivalently $n_r/d_c = n_c/d_r$. Moreover, biregularity implies that

$$\mathbf{1}_{n_r}^\top B = d_c \mathbf{1}_{n_c}^\top, B\mathbf{1}_{n_c} = d_r \mathbf{1}_{n_r}. \tag{7.79}$$

It is easy to verify that $\sqrt{d_r d_c}$ is the largest singular value of B. This motivates the following definition.

Definition 7.28. *Suppose $B \in \{0,1\}^{n_r \times n_c}$ is (d_r, d_c)-biregular. Then the bipartite graph corresponding to B is said to be a **Ramanujan bigraph** if every nonzero singular value σ_i of B other than $\sqrt{d_r d_c} =: \sigma_1$ satisfies the bound*

$$|\sqrt{d_r - 1} - \sqrt{d_c - 1}| \leq \sigma_i \leq \sqrt{d_r - 1} + \sqrt{d_c - 1}, \tag{7.80}$$

or equivalently

$$|\sigma_i^2 - (d_r - 1 + d_c - 1)| \leq 2\sqrt{d_r - 1}\sqrt{d_c - 1}. \tag{7.81}$$

It is obvious that if B is a square matrix so that the corresponding bipartite graph is balanced (and $d_r = d_c$), then the inequality (7.80) reduces to (4.28), because the lower bound is trivially satisfied. Thus when $n_r = n_c$, a Ramanujan bigraph is the same as case 2 of Definition 4.13. The definition of a Ramanujan bigraph in terms of (7.80) is given in [139], while the definition in terms of (7.81) is given in [220].

Note that in [176], a bipartite graph satisfying only the upper bound in (7.80) is referred to as an "irregular" Ramanujan graph. Other possible terms are unbalanced or asymmetric Ramanujan graph.

In the case of d-regular graphs, we have Theorem 4.16, which implies that if we keep d fixed and let n increase, then $2\sqrt{d-1}$ is the best possible bound on the second largest eigenvalue of the adjacency matrix. An analogous bound for bipartite graphs is proved in [109].

Theorem 7.29. *Fix d_r, d_c, and let n_r, n_c approach infinity, subject of course to the constraint that $n_r d_r = n_c d_c$. Let σ_2 denote the second largest singular value of $B \in \{0,1\}^{n_r \times n_c}$. Then*

$$\liminf \sigma_2 \geq \sqrt{d_r - 1} + \sqrt{d_c - 1}. \tag{7.82}$$

[30] Much of the material in this section is a part of the doctoral research of Shantanu Prasad Burnwal at the Indian Institute of Technology Hyderabad. As such, only the theorems are presented, and the reader is referred to [41] for the proofs.

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Therefore the upper bound in (7.80) is the best that one can hope for.

In Section 4.3.2, we have presented two explicit constructions of Ramanujan graphs. We also mentioned some recent results in [175, 176, 177] which show that, for every degree d and number of vertices n, there exists a d-regular Ramanujan bipartite graph with n vertices. Along these lines, it is also shown in [176] that, for every pair of integers (d_r, d_c), both ≥ 3, there exist infinitely many (d_r, d_c)-biregular Ramanujan bigraphs. However, there are very few *explicit* constructions of Ramanujan bigraphs. The papers [16, 17] present some abstract constructions. However, these constructions are not so explicit as those in [172, 181].

Against this background, we now present an explicit construction based on LDPC codes. This construction is presented as [41, Theorem 8] and can be thought of as the first (and thus far the only) explicit construction of a Ramanujan bigraph. Let q be any prime number, and let $P \in \{0,1\}^{q\times q}$ denote the "right shift" permutation. Thus $P_{i,i-1} = 1$ for $i \geq 2$, $P_{q,1} = 1$, and the remaining elements are zero. Next, let l be any integer between 2 and q, and define $B(q,l) \in \{0,1\}^{lq\times q^2}$ as the matrix with the (i,j)th block in $\{0,1\}^{q\times q}$ equal to $P^{(i-1)(j-1)}$. In greater detail, $B(q,l)$ equals

$$B(q,l) = \begin{bmatrix} I_q & I_q & I_q & \dots & I_q \\ I_q & P & P^2 & \dots & P^{q-1} \\ I_q & P^2 & P^4 & \dots & P^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_q & P^{l-1} & P^{2(l-1)} & \dots & P^{(l-1)(q-1)} \end{bmatrix}. \tag{7.83}$$

Note that $P^q = I_q$. Therefore the various powers of P can be computed modulo q.

Theorem 7.30. *The matrix $B(q,l)$ has a singular value of $\sqrt{lq}$, $l(q-1)$ singular values of $\sqrt{q}$, and $l-1$ singular values of 0. Therefore, whenever $2 \leq l \leq q-1$, $B(q,l)$ defines a Ramanujan bigraph. With $l = q$, $B(q,q)$ defines a balanced Ramanujan bipartite graph.*

The proof can be found in [41].

Next we present the rationale for using the biadjacency matrix of a Ramanujan bigraph as the measurement matrix E_Ω. Suppose we could choose $E_\Omega = \mathbf{1}_{n_r\times n_c}$, the matrix of all ones. Then $E_\Omega.X = X$, and we could recover X exactly from the measurements. However, this choice of E_Ω corresponds to measuring *every* element of X, and there would be nothing "compressed" about this sensing. Now suppose that $E_\Omega = B$, the biadjacency matrix of a (d_r, d_c)-biregular graph. Then $\sigma_1 = \sqrt{d_r d_c}$ is the largest singular value of B, with corresponding row and column singular vectors $u_1 = (1/\sqrt{d_r})\mathbf{1}_{n_r}$ and $v_1 = (1/\sqrt{d_c})\mathbf{1}_{n_c}$. Let σ_2 denote the second largest singular value of B. Then

$$B = \sigma_1 u_1 v_1^\top + B_2, \text{ where } \|B_2\|_S \leq \sigma_2,$$

where $\|\cdot\|_S$ denotes the spectral norm of a matrix (i.e., its largest singular value). Using the formulas for u_1 and v_1 and rescaling shows that

$$\sqrt{\frac{n_r n_c}{d_r d_c}} B = \mathbf{1}_{n_r\times n_c} + \sqrt{\frac{n_r n_c}{d_r d_c}} B_2.$$

This formula can be expressed more compactly by defining the constant α as

$$\alpha := \sqrt{\frac{d_r d_c}{n_r n_c}} = \frac{d_r}{n_c} = \frac{d_c}{n_r},$$

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where the various equalities follow from the fact that $n_r d_r = n_c d_c$. One can think of α as the *fraction of elements* of the unknown matrix X that are sampled. Since $\mathbf{1}_{n_r \times n_c}.X = X$, we see that

$$\frac{1}{\alpha} B.X = X + M.X,$$

where $M = (1/\alpha)B_2$. Therefore

$$\left\| \frac{1}{\alpha} B.X - X \right\|_S = \|M.X\|_S. \tag{7.84}$$

Now note that

$$\|M\|_S = \frac{\sigma_2}{\alpha} = \sigma_2 \cdot \sqrt{\frac{n_r n_c}{d_r d_c}} = \frac{\sigma_2}{\sigma_1} \sqrt{n_r n_c}.$$

Therefore, the smaller σ_2 is compared to σ_1, the better the approximation error is between $(1/\alpha)B.X$ and the unknown matrix X.[31] Now, a Ramanujan bigraph is one for which this ratio is as small as possible.

7.5.2 ▪ Some Properties of the Max-Norm and the Nuclear Norm

In this subsection we introduce the so-called max-norm, which provides an alternative to the nuclear norm as the objective function in the matrix completion problem. The advantage of the max-norm is that it is possible to establish bounds between the estimated matrix $\hat{X}$ and the true but unknown matrix X *without* invoking the coherence of the unknown matrix. This is shown in the next subsection. The disadvantage is that, in contrast with nuclear norm minimization, there are no conditions available under which *exact recovery* of the unknown matrix is possible. To lay the foundation for these results, in this subsection we derive several properties of both the max-norm and the nuclear norm.

Recall from Section 1.2, specifically Table 1.1, that the $2 \to \infty$ induced norm of a matrix $A \in \mathbb{R}^{n_r \times n_c}$ is given by

$$\|A\|_{2\to\infty} = \max_{i \in [n_r]} \|a^i\|_2, \tag{7.85}$$

where a^i denotes the ith row of A. We can now define the max-norm.

Definition 7.31. *The **max-norm** of a matrix $A \in \mathbb{R}^{n_r \times n_c}$ is denoted by $\|A\|_M$ and is defined by*

$$\|A\|_M := \min(\|U\|_{2\to\infty} \cdot \|V\|_{2\to\infty}) \text{ s.t. } UV^\top = A, \tag{7.86}$$

where the maximum is taken over all integers l and all $U \in \mathbb{R}^{n_r \times l}, V \in \mathbb{R}^{n_c \times l}$ that satisfy $UV^\top = A$.

Note that the max-norm is also referred to as the γ_2-norm. It is introduced in [162], where several of its properties are analyzed.[32] Apparently the first paper to propose the use of the max-norm to address the matrix completion problem is [222]. An excellent review of the rationale behind the max-norm, including a discussion of why it is a good proxy for the rank of a matrix, is found in [156].

[31] Note that n_r, n_c are the dimensions of the unknown matrix and are therefore fixed.

[32] Note however that in [162], the factorization is taken as $A = XY$ and the quantity to be minimized is taken as $\|X\|_{2\to\infty} \cdot \|Y\|_{1\to 2}$. This is clearly equivalent to (7.86), because, as is evident from Table 1.1, we have that $\|Y\|_{1\to 2} = \|Y^\top\|_{2\to\infty}$.

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Next we give various interpretations of the max-norm and also relate it to the nuclear norm. In order to prove these results, we make heavy use of the concept of dual norms.

Definition 7.32. *Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^{n_r\times n_c}$. Then its **dual norm** on $\mathbb{R}^{n_r\times n_c}$, denoted by $\|\cdot\|^*$, is defined by*

$$\|X\|^* = \max_{\|Y\|\le 1} |\langle X,Y\rangle_F| = \max_{\|Y\|\le 1} \langle X,Y\rangle_F = \max_{\|Y\|=1} |\langle X,Y\rangle_F| = \max_{\|Y\|=1} \langle X,Y\rangle_F. \tag{7.87}$$

For instance, as shown in Theorem 1.19, $\|\cdot\|_N^* = \|\cdot\|_S$. The following properties of dual norms are easy to prove.

1. Suppose $\|\cdot\|_a, \|\cdot\|_b$ satisfy $\|A\|_b = \gamma\|A\|_a$ for all $A \in \mathbb{R}^{n_r\times n_c}$, for some constant γ. Then
$$\|X\|_a^* = \gamma\|X\|_b^*,\ \forall X \in \mathbb{R}^{n_r\times n_c}. \tag{7.88}$$
2. Suppose $\|\cdot\|_a, \|\cdot\|_b$ satisfy $\|A\|_a \le \|A\|_b$ for all $A \in \mathbb{R}^{n_r\times n_c}$. Then
$$\|X\|_b^* \le \|X\|_a^*,\ \forall X \in \mathbb{R}^{n_r\times n_c}. \tag{7.89}$$

Next we give geometric interpretations of what the unit balls in the max-norm and the nuclear norm look like. The discussion is facilitated by some notation. Let $\mathcal{B}(r,\|\cdot\|)$ denote the ball of radius r in the norm $\|\cdot\|$, centered at the origin. Where warranted, the underlying space will be displayed as a subscript on $\mathcal{B}$. Next, let $\mathbb{S}_n$ denote the set of column vectors $\boldsymbol{\alpha} \in \{-1,1\}^n$, the set of "signed" n-vectors. Analogously, define

$$\mathbb{S}^{n_r\times n_c} := \{\boldsymbol{\alpha}\boldsymbol{\beta}^\top : \boldsymbol{\alpha} \in \mathbb{S}_{n_r}, \boldsymbol{\beta} \in \mathbb{S}_{n_c}\},$$

the set of "signed" rank one matrices or order $n_r \times n_c$. With these notational conventions, we begin our analysis of the max-norm and the nuclear norm. The main results are stated in succession, followed by their proofs.

It is evident from the definition of the nuclear norm that

$$\mathcal{B}(1,\|\cdot\|_N) = \mathrm{Conv}(\{uv^\top : \|u\|_2 = 1, \|v\|_2 = 1\} \cup \{0\}).$$

In words, the set $\{A : \|A\|_N = 1\}$ is the convex hull of rank-one matrices of the form $uv^\top$, where both u and v have unit ℓ_2-norm. The set $\{A : \|A\|_N \le 1\}$ is the convex hull of such rank-one matrices together with the zero matrix.

A similar characterization of the unit ball in the max-norm is not available at present. However, one can find both a subset and a superset of $\mathcal{B}(1,\|\cdot\|_M)$. These bounds involve a universal constant known as "Grothendieck's constant," denoted by K_G. The constant K_G is defined in Theorem 7.33 below, where an upper bound for K_G is also provided; see (7.91).

Theorem 7.33. *(Grothendieck's inequality [114].) There exists a universal constant K_G such that, for every set of integers $l, n_r, n_c \ge 1$ and every matrix $A \in \mathbb{R}^{n_r\times n_c}$, we have that*

$$\max_{\|\boldsymbol{\theta}_i\|_2\le 1, \|\boldsymbol{\phi}_j\|_2\le 1} \left|\sum_{i=1}^{n_r}\sum_{j=1}^{n_c} a_{ij}\langle\boldsymbol{\theta}_i,\boldsymbol{\phi}_j\rangle\right| \le K_G \max_{\boldsymbol{\alpha}\in\mathbb{S}_{n_r},\boldsymbol{\beta}\in\mathbb{S}_{n_c}} \left|\sum_{i=1}^{n_r}\sum_{j=1}^{n_c} a_{ij}\alpha_i\beta_j\right|. \tag{7.90}$$

Moreover

$$K_G \le \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1.7821. \tag{7.91}$$

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Note that the maximum on the left side of (7.90) is taken over all unit ℓ_2-norm vectors $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_r}, \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{n_c} \in \mathbb{R}^l$, where the integer l is arbitrary and can be chosen so as to achieve the maximum.

Both sides of (7.90) have obvious interpretations as dual matrix norms. Therefore Grothendieck's inequality is equivalent to the following restatement in terms of matrix norms. Note that the induced matrix norm $\|\cdot\|_{\infty\to 1}$ is defined in (1.26) and $\|\cdot\|^*_{\infty\to 1}$ is its dual.

Theorem 7.34. *(See [223, Corollary 2].) For every pair of integers $n_r, n_c \geq 1$, we have that*

$$\|A\|_{\infty\to 1} \leq \|A\|^*_M \leq K_G \|A\|_{\infty\to 1}, \ \forall A \in \mathbb{R}^{n_r \times n_c}. \tag{7.92}$$

Consequently

$$\mathcal{B}(1, \|\cdot\|^*_{\infty\to 1}) \subseteq \mathcal{B}(1, \|\cdot\|_M) \leq K_G \mathcal{B}(1, \|\cdot\|^*_{\infty\to 1}). \tag{7.93}$$

Since both $\|\cdot\|_{\infty\to 1}$ and $\|\cdot\|^*_M$ are norms on $\mathbb{R}^{n_r \times n_c}$, it is not surprising that they are equivalent. What *is* surprising however is that the constants in (7.92) are independent of the dimensions n_r and n_c.

Theorem 7.34 allows us to state the following result, which states that the unit ball in the max-norm contains the convex hull of all rank-one sign matrices (plus the zero matrix) and is in turn contained in the same set expanded by a factor of K_G.

Theorem 7.35. *We have that*

$$\mathrm{Conv}(\mathbb{S}^{n_r \times n_c} \cup \{0\}) \subseteq \mathcal{B}(1, \|\cdot\|_M) \subseteq K_G \mathrm{Conv}(\mathbb{S}^{n_r \times n_c} \cup \{0\}). \tag{7.94}$$

Now we introduce yet another norm which is a kind of nuclear norm and relate it to the max-norm.

Definition 7.36. *The norm $\|\cdot\|_\nu$ on $\mathbb{R}^{n_r \times n_c}$ is defined by*

$$\|X\|_\nu = \min_{d_i \in \mathbb{R}} \sum_{i \in [l]} |d_i| \text{ s.t. } X = \sum_{i \in [l]} d_i \boldsymbol{\alpha}_i \boldsymbol{\beta}_i^\top \text{ for some } \boldsymbol{\alpha}_i \in \mathbb{S}_{n_r}, \boldsymbol{\beta} \in \mathbb{S}_{n_c}. \tag{7.95}$$

Thus $\|X\|_\nu$ is analogous to the nuclear norm when X is expressed as a sum of rank-one sign matrices. This is brought out in the next theorem, which gives an alternate characterization of the nuclear norm. See also Problem 7.3.

Theorem 7.37. *Define a norm $\|\cdot\|_c$ on $\mathbb{R}^{n_r \times n_c}$ as follows:*

$$\|X\|_a := \min_{d_j} \sum_{j \in [l]} |d_j| \text{ s.t. } X = \sum_{j \in [l]} d_j \theta_j \phi_j^\top, \|\theta_j\|_2 = 1, \|\phi_j\|_2 = 1, \ \forall j \in [l]. \tag{7.96}$$

Then $\|\cdot\|_a = \|\cdot\|_N$.

Theorem 7.38. *$\|\cdot\|_{\infty\to 1}$ is the dual norm of $\|\cdot\|_\nu$. Consequently*

$$\|X\|_M \leq \|X\|_\nu \leq K_G \|X\|_M. \tag{7.97}$$

Next we give yet another alternate characterization of the nuclear norm.

Theorem 7.39. *(See [223, Lemma 1] or [209, Lemma 1].) We have that*

$$\|X\|_N = \min(\|B\|_F \cdot \|C\|_F) \text{ s.t. } X = BC^\top \tag{7.98}$$

$$= \min \frac{1}{2}(\|B\|_F^2 + \|C\|_F^2) \text{ s.t. } X = BC^\top. \tag{7.99}$$

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Theorem 7.39 allows us to relate the nuclear and max norms.

Theorem 7.40. *(See [162, Lemma 3.4].) Let $A \in \mathbb{R}^{n_r \times n_c}$. Then*

$$\frac{1}{\sqrt{n_r n_c}} \|A\|_N \leq \|A\|_M \leq \|A\|_N. \tag{7.100}$$

Now that the main results have all been stated, we proceed to prove them.

***Proof of Theorem* 7.34.** Recall that

$$\|A\|_{\infty \to 1} = \max_{\|y\|_\infty = 1} \|Ay\|_1.$$

The maximum of a convex function over a convex set occurs at an extreme point of the convex set. For the unit ball in the ℓ_∞-norm, the extreme points are the sign vectors. Therefore

$$\|A\|_{\infty \to 1} = \max_{y \in \mathbb{S}_{n_c}} \|Ay\|_1.$$

However, since the dual of the ℓ_1-norm is the ℓ_∞-norm, it is also true that

$$\|Ay\|_1 = \max_{\|x\|_\infty = 1} |\langle x, Ay \rangle| = \max_{\|x\|_\infty = 1} |x^\top Ay|.$$

Again by the same argument

$$\|A\|_{\infty \to 1} = \max_{x \in \mathbb{S}_{n_r}, y \in \mathbb{S}_{n_c}} |x^\top Ay| = \max_{x \in \mathbb{S}_{n_r}, y \in \mathbb{S}_{n_c}} \left| \sum_{i=1}^{n_r} \sum_{j=1}^{n_c} a_{ij} x_i y_j \right|,$$

which is the right side of (7.90) without the coefficient K_G.

Next, from the definition of the max-norm, it follows that

$$\mathcal{B}(1, \|\cdot\|_M) = \{H = \Theta\Phi^\top : \|\Theta\|_{2\to\infty} \leq 1, \|\Phi\|_{2\to\infty} \leq 1\}.$$

This is because if

$$H = BC^\top, \|B\|_{2\to\infty} \cdot \|C\|_{2\to\infty} \leq 1,$$

then we can always scale B, C so that the product is the same but both matrices have norms less than one. To put it another way[33]

$$\mathcal{B}(1, \|\cdot\|_M) = \{H = [\langle \boldsymbol{\theta}^i, \boldsymbol{\phi}^j \rangle], \boldsymbol{\theta}^i, \boldsymbol{\phi}^j \in \mathcal{B}_l(1, \|\cdot\|_2)\}.$$

This is because if we partition Θ, Φ as

$$\Theta = \begin{bmatrix} \boldsymbol{\theta}^1 \\ \vdots \\ \boldsymbol{\theta}^{n_r} \end{bmatrix} \in \mathbb{R}^{n_r \times l}, \Phi = \begin{bmatrix} \boldsymbol{\phi}^1 \\ \vdots \\ \boldsymbol{\phi}^{n_c} \end{bmatrix} \in \mathbb{R}^{n_c \times l},$$

then $\|\Theta\|_{2\to\infty} \leq 1$, $\|\Phi\|_{2\to\infty} \leq 1\}$ is equivalent to $\|\boldsymbol{\theta}^i\|_2 \leq 1$ for $i \in [n_r]$, $\|\boldsymbol{\phi}^j\|_2 \leq 1$ for $j \in [n_c]$.

[33]Note that here and elsewhere, we use the symbol $\langle \boldsymbol{\theta}^i, \boldsymbol{\phi}^i \rangle$ to denote the inner product of *row* vectors. This saves some cumbersome notation.

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Now, from the definition of the dual norm, it follows that

$$\|A\|_M^* = \max_{H \in \mathcal{B}(1,\|\cdot\|_M)} |\langle A, H\rangle_F| = \max_{\|\boldsymbol{\theta}^i\|_2, \|\boldsymbol{\phi}^j\|_2 \le 1} \left| \sum_{i=1}^{n_r} \sum_{j=1}^{n_c} a_{ij} \langle \boldsymbol{\theta}^i, \boldsymbol{\phi}^j \rangle \right|, \tag{7.101}$$

which is the left side of (7.90). Hence Grothendieck's inequality implies that

$$\|A\|_M^* \le K_G \|A\|_{\infty \to 1},$$

which is the right inequality in (7.92). To prove the left inequality in (7.92), choose $l = 1$, and

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n_r} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{n_c} \end{bmatrix}, \alpha_i, \beta_j \in \{-1, 1\}.$$

Then $\|\boldsymbol{\alpha}\|_{2\to\infty} = 1$ and $\|\boldsymbol{\beta}\|_{2\to\infty} = 1$. Therefore $\|\boldsymbol{\alpha}\boldsymbol{\beta}^\top\|_M \le 1$. It follows that

$$\|A\|_M^* \ge |\langle A, \boldsymbol{\alpha}\boldsymbol{\beta}^\top\rangle| = \left| \sum_{i=1}^{n_r} \sum_{j=1}^{n_c} a_{ij} \alpha_i \beta_j \right|.$$

Since this is true for *every* choice of signed vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$, it follows that

$$\|A\|_{\infty \to 1} \le \|A\|_M^*.$$

This completes the proof of (7.92). Now (7.93) is a ready consequence of (7.92) and properties (7.88) and (7.89) of dual norms.

***Proof of Theorem* 7.35.** This consists of showing that the unit ball $\mathcal{B}(1, \|\cdot\|_\infty^*)$ is the convex hull of all signed rank-one matrices and the zero matrix. Note that

$$\|X\|_\infty^* \le 1 \iff |\langle A, X\rangle_F| \le 1, \ \forall A \in \mathcal{B}(1, \|\cdot\|_\infty).$$

Equivalently

$$\|X\|_\infty^* \le 1 \iff |\langle A, X\rangle_F| \le 1 \text{ whenever } |\langle A, Y\rangle_F| \le 1, \ \forall Y \in \mathbb{S}^{n_r \times n_c}.$$

This completes the proof.

***Proof of Theorem* 7.37.** Let $X = U\Sigma V^\top$ be a reduced SVD of X, so that $U \in \mathbb{R}^{n_r \times r}$, $V \in \mathbb{R}^{n_c \times r}$ have orthonormal columns, and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$. Therefore $X = \sum_{i \in [r]} \sigma_i u_i v_i^\top$, and moreover, the matrices $\mathbf{u}_i v_i^\top$ are orthonormal under the Frobenius inner product. Therefore $\sigma_i = \langle X, u_i v_i^\top\rangle_F$ for each $i \in [r]$. Now suppose that

$$X = \sum_{j \in [l]} d_j \theta_j \phi_j^\top$$

for some integer l, real numbers d_j, and unit ℓ_2-norm vectors θ_j, ϕ_j. Then

$$\sigma_i = \langle u_x v_i^\top, X\rangle_F = \sum_{j \in [l]} d_j \langle u_i v_i^\top, \theta_j \phi_j^\top\rangle_F = \sum_{j \in [l]} d_j \langle u_i, \theta_j\rangle \cdot \langle v_i, \phi_j\rangle.$$

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Therefore

$$\sum_{i\in[r]} \sigma_i = \sum_{j\in[l]} d_j \sum_{i\in[r]} \langle u_i, \theta_j\rangle \cdot \langle v_i, \phi_j\rangle.$$

Define

$$c_j := \sum_{i\in[r]} \langle u_i, \theta_j\rangle \cdot \langle v_i, \phi_j\rangle.$$

Then by Schwarz's inequality

$$|c_j|^2 \leq \left(\sum_{i\in[r]} \langle u_i, \theta_j\rangle^2\right)\left(\sum_{i\in[r]} \langle v_i, \phi_j\rangle^2\right) \leq \|\theta_j\|_2^2 \cdot \|\phi_j\|_2^2 \leq 1,$$

where we use the fact that both $\{u_i\}_{i=1}^r$ and $\{v_i\}_{i=1}^r$ are orthonormal systems, and both θ_j and ϕ_j are unit vectors. Thus

$$\sum_{i\in[r]} \sigma_i = \sum_{j\in[l]} d_j c_j = \left|\sum_{j\in[l]} d_j c_j\right| \leq \sum_{j\in[l]} |d_j| \cdot |c_j| \leq \sum_{j\in[l]} |d_j|.$$

Taking the minimum of the right side with respect to all representations of X in the form (7.96) shows that $\|X\|_N \leq \|X\|_a$.

In the other direction, choose $l = r$, $d_i = \sigma_i$, $\theta_i = u_i$, and $\phi_i = v_i$. This shows that $\|X\|_a \leq \|X\|_N$.

***Proof of Theorem* 7.38.** By definition, the dual norm of $\|\cdot\|_\nu$ is given by

$$\|A\|_\nu^* = \max_{\|X\|_\nu \leq 1} |\langle X, A\rangle_F|.$$

It is now shown that, for all $A \in \mathbb{R}^{n_r \times n_c}$, two statements are true:

$$|\langle X, A\rangle_F| \leq \|A\|_{\infty\to 1} \text{ whenever } \|X\|_\nu \leq 1, \tag{7.102}$$

and

$$\exists X \text{ with } \|X\|_\nu \leq 1 \text{ s.t. } \langle X, A\rangle_F = \|A\|_{\infty\to 1}. \tag{7.103}$$

These two relationships suffice to show that $\|\cdot\|_{\infty\to 1}$ is the dual norm of $\|\cdot\|_\nu$.

To prove these two claims, recall from (7.93) that

$$\|A\|_{\infty\to 1} = \max_{\boldsymbol{\alpha}\in\mathbb{S}_{n_r}, \boldsymbol{\beta}\in\mathbb{S}_{n_c}} |\boldsymbol{\alpha}^\top A \boldsymbol{\beta}|.$$

Now suppose $\|X\|_\nu \leq 1$, and choose l, $d_1, \ldots, d_l$, $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_l \in \mathbb{S}_{n_r}$, and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_l \in \mathbb{S}_{n_c}$ such that

$$X = \sum_{i\in[l]} d_i \boldsymbol{\alpha}_i \boldsymbol{\beta}_i^\top, \sum_{i\in[l]} |d_i| \leq 1.$$

Then

$$\begin{aligned}|\langle X, A\rangle_F| &= \left|\sum_{i\in[l]} d_i \langle \boldsymbol{\alpha}_i \boldsymbol{\beta}_i^\top, A\rangle\right| \leq \sum_{i\in[l]} |d_i| \cdot |\langle \boldsymbol{\alpha}_i \boldsymbol{\beta}_i^\top, A\rangle| \\ &\leq \left(\sum_{i\in[l]} |d_i|\right) \cdot \|A\|_{\infty\to 1} \leq \|A\|_{\infty\to 1}.\end{aligned}$$

This establishes (7.102).

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To establish (7.103), choose $\boldsymbol{\alpha} \in \mathbb{S}_{n_r}, \boldsymbol{\beta} \in \mathbb{S}_{n_c}$ such that $\boldsymbol{\alpha}^\top A\boldsymbol{\beta} = \|A\|_{\infty\to 1}$.[34] Let $l = 1$ and define $X = \boldsymbol{\alpha}\boldsymbol{\beta}^\top$. Then $\|X\|_\nu \le 1$, yet

$$\langle X, A\rangle_F = \boldsymbol{\alpha}^\top A\boldsymbol{\beta} = \|A\|_{\infty\to 1}.$$

This completes the proof that $\|\cdot\|_\nu$ is the dual norm of $\|\cdot\|_{\infty\to 1}$.

Now (7.97) follows from (7.92) and the properties of dual norms.

***Proof of Theorem* 7.39.** Suppose $X \in \mathbb{R}^{n_r\times n_c}$ has rank r, and let $X = U\Sigma V^\top$ be its reduced SVD, so that $U \in \mathbb{R}^{n_r\times r}$ and $V \in \mathbb{R}^{n_c\times r}$. Now suppose that $X = BC^\top$ for some $B \in \mathbb{R}^{n_r\times l}$, $C \in \mathbb{R}^{n_c\times l}$. Then, using the facts that $U^\top U = V^\top V = I_r$, we get

$$\Sigma = U^\top XV = U^\top BC^\top V = (U^\top B)(V^\top C) = GH^\top,$$

where $G = U^\top B \in \mathbb{R}^{r\times l}$, $H = V^\top C \in \mathbb{R}^{r\times l}$.

Now it is established that $\|G\|_F \le \|B\|_F$, $\|H\|_F \le \|C\|_F$. Let g_j, b_j denote the jth columns of G, B, respectively, and observe that $g_j = U^\top b_j$. Further, $U^\top U = I_r$ implies that $\|U^\top\|_S = 1$, so that $\|g_j\|_2 \le \|b_j\|_2$ for all $j \in [l]$. Finally

$$\|G\|_F^2 = \sum_{j\in[l]} \|g_j\|_2^2 \le \sum_{j\in[l]} \|b_j\|_2^2 = \|B\|_F^2.$$

By entirely analogous reasoning, it follows that $\|H\|_F \le \|C\|_F$.

Next, it is shown that if $\Sigma = GH^\top$ for some $G, H \in \mathbb{R}^{r\times l}$, then

$$\|G\|_F^2 + \|H\|_F^2 \ge 2\|\boldsymbol{\sigma}\|_1, \tag{7.104}$$

$$\|G\|_F + \|H\|_F \ge \|\boldsymbol{\sigma}\|_1, \tag{7.105}$$

where $\boldsymbol{\sigma} \in \mathbb{R}_+^r$ is the vector of singular values of X. Let us begin with (7.104). The relation $\Sigma = GH^\top$ implies that $\langle g^i, h^i\rangle = \sigma_i$, where g^i, h^i denote the ith rows of G, H, respectively. The same relation also implies that $\langle g^i, h^j\rangle = 0$ whenever $i \ne j$, but we do not make any use of this. By Schwarz's inequality, it follows that

$$\|g^i\|_2 \cdot \|h^i\|_2 \ge \langle g^i, h^i\rangle = \sigma_i.$$

For convenience let $\alpha_i = \|g^i\|_2$, $\beta_i = \|h^i\|_2$. Now a simple exercise using Lagrange multipliers shows that

$$\min(\alpha_i^2 + \beta_i^2) \text{ s.t. } \alpha_i\beta_i \ge \sigma_i = 2\sigma_i,$$

corresponding to the choice $\alpha_i = \beta_i = \sqrt{\sigma_i}$. In other words

$$\langle g^i, h^i\rangle = \sigma_i \implies \|g^i\|_2^2 + \|h^i\|_2^2 \ge 2\sigma_i.$$

Therefore $GH^\top = \Sigma$ implies that

$$\|G\|_F^2 + \|H\|_F^2 = \sum_{i\in[r]} (\|g^i\|_2^2 + \|h^i\|_2^2) \ge 2\sum_{i\in[r]} \sigma_i = 2\|X\|_N.$$

It has already been established that if $X = BC^\top$, then $\|G\|_F \le \|B\|_F$, $\|H\|_F \le \|C\|_F$. Therefore

$$X = BC^\top \implies \|B\|_F^2 + \|C\|_F^2 \ge 2\|X\|_N.$$

[34] Note that it is NP-hard to actually *find* such a pair $\boldsymbol{\alpha}, \boldsymbol{\beta}$. But such a pair surely exists.

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To prove (7.105), observe that

$$\|G\|_F^2 \cdot \|H\|_F^2 = \left(\sum_{i\in[r]} \alpha_i^2\right)\left(\sum_{j\in[r]} \beta_j^2\right),$$

where as before $\alpha_i = \|g^i\|_2$, $\beta_j = \|h^j\|_2$. It is again a simple exercise using Lagrange multipliers to show that

$$\min \left(\sum_{i\in[r]} \alpha_i^2\right)\left(\sum_{j\in[r]} \beta_j^2\right) \text{ s.t. } \alpha_i\beta_j \geq \sigma_i = \|\boldsymbol{\sigma}\|_1^2,$$

or equivalently

$$\min \left(\sum_{i\in[r]} \alpha_i^2\right)^{1/2}\left(\sum_{j\in[r]} \beta_j^2\right)^{1/2} \text{ s.t. } \alpha_i\beta_j \geq \sigma_i = \|\boldsymbol{\sigma}\|_1.$$

Thus

$$X = BC^\top \implies \|B\|_F \cdot \|C\|_F \geq \|G\|_F \cdot \|H\|_F \geq \|\boldsymbol{\sigma}\|_1.$$

To complete the proof, for a given matrix X, we select B, C such that the inequalities in (7.104) and (7.105) become equalities. Let $X = U\Sigma V^\top$ be the reduced SVD of X, and choose $B = U\Sigma^{1/2}$, $C = V\Sigma^{1/2}$. Then, because each column of U and of V has unit ℓ_2-norm, it follows that $\|B\|_F^2 = \|C\|_F^2 = \|\boldsymbol{\sigma}\|_1$.

***Proof of Theorem* 7.40.** Suppose $B \in \mathbb{R}^{n_r\times l}$. Then

$$\|B\|_{2\to\infty} = \max_{i\in[n_r]} \|b^i\|_2 = \|\boldsymbol{\theta}\|_\infty,$$

where $\boldsymbol{\theta} = [\|b^1\|_2 \ldots \|b^{n_r}\|_2]$, while

$$\|B\|_F = \|\boldsymbol{\theta}\|_2 \geq \|\boldsymbol{\theta}\|_\infty.$$

Therefore, if $X = BC^\top$, then

$$\|B\|_{2\to\infty} \cdot \|C\|_{2\to\infty} \leq \|B\|_F \cdot \|C\|_F.$$

The definition of the max-norm and (7.99) now lead to

$$\min_{BC^\top = X} \|B\|_{2\to\infty} \cdot \|C\|_{2\to\infty} \leq \min_{BC^\top = X} \|B\|_F \cdot \|C\|_F,$$

which is the right inequality in (7.100).

To prove the left inequality in (7.100), we prove instead that

$$\|X\|_M^* \leq \sqrt{n_r n_c}\|X\|_S, \ \forall X \in \mathbb{R}^{n_r\times n_c}. \tag{7.106}$$

Because $\|\cdot\|_S$ and $\|\cdot\|_N$ are dual norms of each other, (7.106) coupled with (7.88) and (7.89) imply the left inequality in (7.100). To prove (7.106), we prove instead the following equivalent statement:

$$\|A\|_M \leq 1 \implies \langle X, A\rangle_F \leq \sqrt{n_r n_c}\|X\|_S. \tag{7.107}$$

Accordingly, suppose that $\|A\|_M \leq 1$, and choose $U \in \mathbb{R}^{n_r\times l}$, $V \in \mathbb{R}^{n_c\times l}$ such that

$$A = UV^\top, \|U\|_{2\to\infty} \leq 1, \|V\|_{2\to\infty} \leq 1.$$

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Then

$$\begin{aligned}\langle X, A\rangle_F = \langle X, UV^\top\rangle_F &= \sum_{j\in[l]} \langle X, u_j v_j^\top\rangle \\ &= \sum_{j\in[l]} u_j^\top X v_j \le \|X\|_S \sum_{j\in[l]} \|u_j\|_2 \cdot \|v_j\|_2 \\ &\le \|X\|_S \left(\sum_{j\in[l]} \|u_j\|_2^2\right)^{1/2} \left(\sum_{j\in[l]} \|v_j\|_2^2\right)^{1/2} \\ &= \|X\|_S \left(\sum_{i\in[n_r]} \|u^i\|_2^2\right)^{1/2} \left(\sum_{i\in[n_c]} \|v^i\|_2^2\right)^{1/2} \\ &\le \sqrt{n_r n_c}\|X\|_S,\end{aligned}$$

because $\|u^i\|_2^2 \le 1$ for all i, and $\|v^i\|_2^2 \le 1$ for all i. This establishes (7.107).

Problem 7.3. *By mimicking the proof of Theorem 7.38, prove the following: Suppose $p, q \in [1, \infty]$, and define the associated norm*

$$\|X\|_{\nu,p,q} := \min_{d_j} \sum_{j\in[l]} |d_j| \text{ s.t. } X = \sum_{j\in[l]} d_j \theta_j \phi_j^\top, \|\theta_j\|_p = 1, \|\phi_j\|_q^* = 1, \ \forall j \in [l].$$

Show that $\|\cdot\|_{\nu,p,q}$ is the dual of the induced matrix norm $\|\cdot\|_{p\to q}$.

7.5.3 ▪ Error Bounds Using the Max-Norm

In this section we present some bounds on the reconstruction error between the true matrix X and the matrix $\hat{X}$ recovered using max-norm minimization. In the literature to date, such estimates have been given for *square* matrices. The results presented here extend such estimates to *rectangular* matrices and also improve upon earlier bounds. The contents of this subsection are taken from [41].

Let us recall the problem formulation. There is an unknown matrix $X \in \mathbb{R}^{n_r \times n_c}$ of rank r or less, where r is a known upper bound. There is also a measurement set $\Omega \subseteq [n_r] \times [n_c]$ and the associated measurement matrix $E_\Omega \in \{0,1\}^{n_r \times n_c}$ defined by

$$(E_\Omega)_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \Omega, \\ 0 & \text{if } (i,j) \notin \Omega. \end{cases}$$

The data consists of the values $X_{ij}, (i,j) \in \Omega$, or equivalently the Hadamard product $E_\Omega.X$. The objective is to construct an approximation to X.

Throughout this subsection, it is assumed that the matrix E_Ω is the biadjacency matrix of a biregular graph with row degree d_r and column degree d_c. Therefore $\mathbf{1}_{n_r}^\top E_\Omega = d_c \mathbf{1}_{n_c}^\top$ and $E_\Omega \mathbf{1}_{n_c} = d_r \mathbf{1}_{n_r}$. Moreover, the largest singular value of E_Ω is $\sigma_1 := \sqrt{d_r d_c}$, with associated row and column singular vectors $\mathbf{1}_{n_r}$ and $\mathbf{1}_{n_c}$, respectively.

In Theorem 7.41, an estimate is constructed via max-norm minimization, that is,

$$\hat{X}_M = \arg\min_Z \|Z\|_M \text{ s.t. } E_\Omega.Z = E_\Omega.X, \tag{7.108}$$

Theorem 7.41 is an extension of [131, Theorem 2] to rectangular matrices.

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Theorem 7.41. *Suppose E_Ω is (d_r, d_c)-biregular, and let σ_2 denote the second largest singular value of E_Ω (and of course $s_1 = \sqrt{d_r d_c}$). Suppose $X \in \mathbb{R}^{n_r \times n_c}$, and define*

$$\hat{X} := \underset{Z \in \mathbb{R}^{n_r \times n_c}}{\arg\min} \|Z\|_M \text{ s.t. } E_\Omega.Z = E_\Omega.X. \tag{7.109}$$

Then

$$\frac{1}{n_r n_c}\|\hat{X} - X\|_F^2 \leq 4K_G \frac{\sigma_2}{\sigma_1}\|X\|_M^2. \tag{7.110}$$

Note that there are no assumptions regarding either the coherence or the rank of the unknown matrix X. Also, the bound given in (7.110) improves the bound in [131, Theorem 2] by a factor of two, in addition to generalizing the result to rectangular matrices.

Now we present a proof of Theorem 7.41. A key part of the proof, known as the "expander mixing lemma" in [136, Lemma 2.5], is of independent interest and is stated here as Theorem 7.35 below. This theorem generalizes the result in [136, Lemma 2.5] in two ways. First, the bound is tighter, and second, the result is applicable also to rectangular graphs. Note that in [136], the expander mixing lemma is credited to [7].

Theorem 7.42. *(Expander mixing lemma for biregular graphs.) Suppose $A \in \{0,1\}^{n_r \times n_c}$ is (d_r, d_c)-biregular. Let $|\mathcal{E}| = n_r d_r = n_c d_c$ denote the total number of edges, and let σ_2 denote the second largest singular value of A. Finally, for any subset S of right vertices and any subset T of left vertices, let $|\mathcal{E}(S,T)|$ denote the number of edges connecting these subsets. Then*

$$\left|\frac{|\mathcal{E}(S,T)|}{|\mathcal{E}|} - \frac{|S|}{n_r}\frac{|T|}{n_c}\right| \leq \frac{\sigma_2}{|\mathcal{E}|}\sqrt{|S|\cdot|T|}\sqrt{\left(1 - \frac{|S|}{n_r}\right)\left(1 - \frac{|T|}{n_c}\right)} \tag{7.111}$$

$$= \frac{\sigma_2}{|\mathcal{E}|}\sqrt{\frac{|S|\cdot|S^c|\cdot|T|\cdot|T^c|}{n_r n_c}} \tag{7.112}$$

$$\leq \frac{\sigma_2}{4\sigma_1}. \tag{7.113}$$

Remarks: For d-regular graphs with n vertices, the bound in (7.111) becomes

$$\left|\frac{|\mathcal{E}(S,T)|}{|\mathcal{E}|} - \frac{|S|\cdot|T|}{n^2}\right| \leq \frac{\sigma_2}{dn}\sqrt{|S|\cdot|T|}\sqrt{\left(1 - \frac{|S|}{n}\right)\left(1 - \frac{|T|}{n}\right)}. \tag{7.114}$$

This can be contrasted with the bound in [136, Lemma 2.5], which, after a little manipulation, is

$$\left|\frac{|\mathcal{E}(S,T)|}{|\mathcal{E}|} - \frac{|S|\cdot|T|}{n^2}\right| \leq \frac{\sigma_2}{dn}\sqrt{|S|\cdot|T|}. \tag{7.115}$$

Clearly, the bound in (7.114) is smaller than that in (7.115), especially as $|S|, |T|$ become large. In the limit, if $|S| = |T| = n$, the left side of (7.114) (or (7.115)) becomes zero, as does the bound on the right side of (7.114), but not the bound in (7.115). Therefore (7.111) is an improvement over the existing bound. Equation (7.112) is the same as (7.111) but written in a form that is symmetric in S and S^c, as well as T and T^c. Finally, note that the right side of (7.113) can also be written as $\sigma_2/(4\sigma_1)$.

Proof. Let r denote the rank of A, and write

$$A = U\Sigma V^\top = \sum_{i \in [r]} \sigma_i u_i v_i^\top.$$

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The biregularity assumption implies that

$$\nu_1 = \sqrt{d_r d_c}, u_1 = (1/\sqrt{n_r})\mathbf{1}_{n_r}, v_1 = (1/\sqrt{n_c})\mathbf{1}_{n_c}.$$

Now let $\mathbf{1}_S \in \{0,1\}^{n_r}$, $\mathbf{1}_T \in \{0,1\}^{n_c}$ denote the indicator vectors of the sets S and T, respectively, and note that $\mathcal{E}(S,T) = \mathbf{1}_S^\top A \mathbf{1}_T$. Carry out an orthogonal expansion

$$\mathbf{1}_S = \sum_{i\in[r]} \alpha_i u_i + a, \mathbf{1}_T = \sum_{i\in[r]} \beta_i v_i + b,$$

where $a^\top A = \mathbf{0}, Ab = \mathbf{0}$. Also observe that

$$\alpha_1 = \langle \mathbf{1}_S, u_1 \rangle = \frac{\langle \mathbf{1}_S, \mathbf{1}_{n_r} \rangle}{\sqrt{n_r}} = \frac{|S|}{\sqrt{n_r}}, \beta_1 = \frac{|T|}{\sqrt{n_c}}.$$

Next

$$\begin{aligned} |\mathcal{E}(S,T)| = \mathbf{1}_S^\top A \mathbf{1}_T &= \sum_{i\in[r]} \sigma_i \alpha_i \beta_i \\ &= \nu_1 \alpha_1 \beta_1 + \sum_{i=2}^{r} \sigma_i \alpha_i \beta_i. \end{aligned} \tag{7.116}$$

Therefore

$$|\mathcal{E}(S,T)| - \nu_1 \alpha_1 \beta_1 = \sum_{i=2}^{r} \sigma_i \alpha_i \beta_i.$$

Next, by Schwarz's inequality, it follows that

$$\left| \sum_{i=2}^{r} \sigma_i \alpha_i \beta_i \right| \leq \left(\sum_{i=2}^{r} \sigma_i^2 \alpha_i^2 \right)^{1/2} \left(\sum_{i=2}^{r} \beta_i^2 \right)^{1/2} \leq \sigma_2 \left(\sum_{i=2}^{r} \alpha_i^2 \right)^{1/2} \left(\sum_{i=2}^{r} \beta_i^2 \right)^{1/2}.$$

Note that

$$\sum_{i=2}^{r} \alpha_i^2 = \|\mathbf{1}_S\|_2^2 - \alpha_1^2 = |S| - \frac{|S|^2}{n_r} = |S| \left(1 - \frac{|S|}{n_r} \right),$$

and similarly

$$\sum_{i=2}^{r} \beta_i^2 = |T| \left(1 - \frac{|T|}{n_c} \right).$$

This implies that

$$\left| \sum_{i=2}^{r} \sigma_i \alpha_i \beta_i \right| \leq \sigma_2 \sqrt{|S| \cdot |T|} \left(1 - \frac{|S|}{n_r} \right)^{1/2} \left(1 - \frac{|T|}{n_c} \right)^{1/2}. \tag{7.117}$$

Substituting this into (7.116) gives

$$\left| \frac{|\mathcal{E}(S,T)|}{|\mathcal{E}|} - \frac{\nu_1 \alpha_1 \beta_1}{|\mathcal{E}|} \right| \leq \sigma_2 \sqrt{|S| \cdot |T|} \left(1 - \frac{|S|}{n_r} \right)^{1/2} \left(1 - \frac{|T|}{n_c} \right)^{1/2}.$$

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The proof is concluded by showing that

$$\frac{\nu_1\alpha_1\beta_1}{|\mathcal{E}|} = \frac{|S|}{n_r}\frac{|T|}{n_c}.$$

Now

$$\nu_1\alpha_1\beta_1 = \sqrt{\frac{d_r d_c}{n_r n_c}}|S|\cdot|T|,$$

while

$$|\mathcal{E}| = n_r d_r = n_c d_c = \sqrt{n_r d_r n_c d_c}.$$

Therefore

$$\frac{\nu_1\alpha_1\beta_1}{|\mathcal{E}|} = \frac{1}{\sqrt{n_r d_r n_c d_c}}\sqrt{\frac{d_r d_c}{n_r n_c}}|S|\cdot|T| = \frac{|S|}{n_r}\frac{|T|}{n_c}.$$

This completes the proof of (7.111).

To prove (7.112), observe that $|S^c| = n_r - |S|$ and $|T^c| = n_c - |T|$. Substituting these into (7.111) gives (7.112).

To prove (7.113), observe that $|S| + |S^c| = n_r$ and $|T| + |T^c| = n_c$. With these constraints, we infer that

$$|S|\cdot|S^c| \le n_r^2/4, |T|\cdot|T^c| \le n_c^2/4.$$

Now write $|\mathcal{E}| = \sqrt{d_r d_c n_r n_c}$. Substituting these relations into (7.112) gives (7.113).

The proof of Theorem 7.41 makes use of the following estimate, which might be of independent interest.

Theorem 7.43. *Suppose E_Ω is (d_r, d_c)-biregular, and let σ_2 denote the second largest singular value of E_Ω. Suppose $R \in \mathbb{R}^{n_r\times n_c}$. Then*

$$\left|\frac{1}{n_r n_c}\sum_{(i,j)\in[n_r]\times[n_c]} R_{ij} - \frac{1}{|\mathcal{E}|}\sum_{(i,j)\in E_\Omega} R_{ij}\right| \le \frac{\sigma_2}{\sigma_1}\|R\|_\nu \tag{7.118}$$

$$\le \frac{\sigma_2}{\sigma_1}K_G\|R\|_M. \tag{7.119}$$

The theorem states that the average value of the elements of an arbitrary matrix R is fairly well approximated by its average over the vertices of a biregular graph. The bound in (7.119) improves upon the bound in [131, Theorem 8] by a factor of two, in addition to generalizing the result to rectangular matrices.

***Proof of Theorem* 7.43.** To simplify notation, we use the shorthand

$$\sum_{i,j} R_{ij} = \sum_{(i,j)\in[n_r]\times[n_c]} R_{ij},$$

and we also use E instead of E_Ω.

We begin by proving the theorem for rank-one sign matrices of the form $M = \boldsymbol{\alpha}\boldsymbol{\beta}^\top$, where both $\boldsymbol{\alpha}, \boldsymbol{\beta}$ have only ± 1 as elements. Define

$$A := \{i \in [n_r] : \alpha_i = 1\}, B := \{j \in [n_c] : \beta_j = 1\}.$$

Then it is clear that $\boldsymbol{\alpha} = \mathbf{1}_A - \mathbf{1}_{A^c}$, $\boldsymbol{\beta} = \mathbf{1}_B - \mathbf{1}_{B^c}$. Now define $J = \mathbf{1}_{n_r}\mathbf{1}_{n_c}^\top$ (the matrix of all ones), and observe that $\mathbf{1}_{n_r} = \mathbf{1}_A + \mathbf{1}_{A^c}$, $\mathbf{1}_{n_c} = \mathbf{1}_B + \mathbf{1}_{B^c}$. Now define $M' = (1/2)(M + J)$,

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$M = 2M' - J$. Then some elementary algebra gives $M' = \mathbf{1}_A \mathbf{1}_B^\top + \mathbf{1}_{A^c}\mathbf{1}_{B^c}^\top$. We can now apply the expander mixing lemma Theorem 7.42 to such a matrix, which gives

$$\begin{aligned}
\left| \frac{1}{n_r n_c} \sum_{i,j} M_{ij} - \frac{1}{|\mathcal{E}|} \sum_{(i,j)\in\mathcal{E}} M_{ij} \right| &= \left| \frac{1}{n_r n_c} \sum_{i,j} (2M'_{ij} - 1) - \frac{1}{|\mathcal{E}|} \sum_{(i,j)\in\mathcal{E}} (2M'_{ij} - 1) \right| \\
&= 2 \left| \frac{1}{n_r n_c} \sum_{i,j} M'_{ij} - \frac{1}{|\mathcal{E}|} \sum_{(i,j)\in\mathcal{E}} M'_{ij} \right| \\
&= 2 \left| \frac{|A| \cdot |B|}{n_r n_c} + \frac{|A^c| \cdot |B^c|}{n_r n_c} - \frac{|\mathcal{E}(A,B)|}{|\mathcal{E}|} - \frac{|\mathcal{E}(A^c,B^c)|}{|\mathcal{E}|} \right| \\
&\leq 2 \left[\left| \frac{|A| \cdot |B|}{n_r n_c} - \frac{|\mathcal{E}(A,B)|}{|\mathcal{E}|} \right| + \left| \frac{|A^c| \cdot |B^c|}{n_r n_c} - \frac{|\mathcal{E}(A^c,B^c)|}{|\mathcal{E}|} \right| \right] \\
&\leq \frac{\sigma_2}{\sigma_1},
\end{aligned}$$

where the last step follows from (7.113).

Now let R be an arbitrary matrix, and express R as $\sum_l \delta_l M_l$, where each M_l is a rank-one sign matrix, and $\sum_l |\delta_l| = \|R\|_\nu$. To simplify notation, let us denote the two averages by

$$\bar{R} = \frac{1}{n_r n_c} \sum_{i,j} R_{ij}, \bar{R}_E = \frac{1}{|\mathcal{E}|} \sum_{(i,j)} R_{ij}.$$

Then a routine application of the triangle inequality shows that

$$|\bar{R} - \bar{R}_E| \leq \sum_l |\delta_l| |\bar{M}_l - \bar{M}_{lE}| \leq \frac{\sigma_2}{\sigma_1} \left(\sum_l |\delta_l| \right) = \frac{\sigma_2}{\sigma_1} \|R\|_\nu,$$

which is (7.118). Now (7.119) follows from (7.97).

The proof of Theorem 7.41 makes use of the fact that the max-norm is multiplicative under the Hadamard product. See, for example, [156, Theorem 17].

***Proof of Theorem* 7.41.** Apply Theorem 7.43 to the matrix $R = (\hat{X} - X).(\hat{X} - X)$, so that $R_{ij} = (\hat{x}_{ij} - x_{ij})^2$. Then $R_{ij} = 0$ whenever $(i,j) \in E_\Omega$ because of the constraint in (7.109). Therefore (7.119) gives

$$\frac{1}{n_r n_c} \|\hat{X} - X\|_F^2 \leq \frac{\sigma_2}{\sigma_1} K_G \|R\|_M. \tag{7.120}$$

Next, the multiplicativity of $\|\cdot\|_M$ implies that

$$\|R\|_M = \|\hat{X} - X\|_M^2 \leq (\|\hat{X}\|_M + \|X\|_M)^2 \leq 4\|X\|_M^2,$$

where we use the fact that $\|\hat{X}\|_M \leq \|X\|_M$. Substituting this into (7.120) gives (7.110).

7.5.4 ▪ Matrix Completion via Nuclear Norm Minimization

In this subsection we present two distinct sets of results. First, it is shown that a suitably scaled version of the measured matrix, without any optimization at all, is a reasonable approximation to the unknown matrix. Second, it is shown that, under suitable conditions, the unknown matrix

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can be recovered exactly via nuclear norm minimization. Only the theorems are stated, and the reader is referred to [41] for the proofs.

Theorem 7.44. *(See [41, Theorem 5].) Suppose the sampling set Ω comes from a (d_r, d_c)-regular bipartite graph, and $\sigma_2 = \sigma_2(E_\Omega)$ denotes the magnitude of the second largest singular value of E_Ω. (Of course $\sigma_1 = \sqrt{d_r d_c}$ is the largest singular value.) Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is a matrix of rank r or less, and let μ_0 denote its coherence as defined in Definition 7.23. Then*

$$\left\| \frac{1}{\alpha} E_\Omega . X - X \right\|_S \leq \frac{\sigma_2}{\sigma_1} \mu_0 r \|X\|_S, \tag{7.121}$$

where $\| \cdot \|_S$ denotes the spectral norm (largest singular value) of a matrix.

Remark: Observe that the bound in (7.121) is a product of two terms: σ_2/σ_1 which depends on the measurement matrix E_Ω, and $\mu_0 r \|X\|_S$ which depends on the unknown matrix X.

Corollary 7.45. *Suppose the sampling set Ω comes from a (d_r, d_c)-regular asymmetric Ramanujan graph, Then*

$$\left\| \frac{1}{\alpha} E_\Omega . X - X \right\| \leq \mu_0 r \left| \frac{1}{\sqrt{d_r}} + \frac{1}{\sqrt{d_c}} \right| \cdot \|X\|_S. \tag{7.122}$$

Now a bit of notation is introduced to state the result on exact recovery via nuclear norm minimization. Assume that there is a constant θ such that

$$\| \sum_{k \in S} \frac{n_r}{d_c} (U^{k\top} U^k) - I_r \|_S \leq \theta, \ \forall S \subseteq [n_r], |S| = d_c, \tag{7.123}$$

$$\| \sum_{k \in S} \frac{n_c}{d_r} (V^{k\top} V^k) - I_r \|_S \leq \theta, \ \forall S \subseteq [n_c], |S| = d_r, \tag{7.124}$$

where $U^{k\top}$ is shorthand for $(U^k)^\top$. Note that if $S = [n_r]$, then $n_r/|S| = 1$, and

$$\sum_{k \in [n_r]} U^{k\top} U^k = I_r.$$

Therefore it is reasonable to assume that (7.123) and (7.124) hold for sufficiently large d_r, d_c.

Theorem 7.46. *(See [41, Theorem 7].) Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is a matrix of rank r or less, and let μ_0, θ be as defined above. Suppose $E_\Omega \in \{0,1\}^{n_r \times n_c}$ is a biadjacency matrix of a (d_r, d_c) biregular graph Ω, and let σ_2 denote the second largest singular value of matrix E_Ω. Define*

$$\phi = \frac{\sigma_2}{\sigma_1} \mu_0 r, \tag{7.125}$$

and suppose that

$$\theta + \phi < 1/2, \tag{7.126}$$

$$\left(1 + \frac{4}{3}\sqrt{\frac{r}{2}}\right) \phi + \theta < 1. \tag{7.127}$$

Then X is the unique minimum of (7.65).

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Notes and References

Some important contributions to matrix recovery using rank-one measurements, and probabilistic methods for matrix completion, are just mentioned in passing in the chapter. The interested reader is referred to [50] for an analysis of rank-one measurements, to [54, 207] for detailed analyses of matrix completion using probabilistic sampling, and to [147, 148] for convergence analysis of the OptSpace algorithm.

Inequality (7.110) can be interpreted as saying the following: Suppose $\hat{X}$ is obtained using max-norm minimization, and elements are drawn from both X and $\hat{X}$ at random, uniformly distributed over the set $[n_r] \times [n_c]$. Then the expected value of the error squared is bounded by the right side of (7.110). While this interpretation is valid, it also negates one of the chief advantages of max-norm minimization, namely that bounds similar to (7.110) can be derived even with *nonuniform* probabilities. Specifically, suppose p_{ij} is a probability distribution over the set $[n_r] \times [n_c]$. Then it is possible to find upper bounds on the expected value of the error

$$\sum_{i,j} p_{ij}(\hat{X}_{ij} - X_{ij})^2.$$

See [131, Theorem 4]. It is shown in [51] that, in the case of nonuniform sampling, nuclear norm minimization does not perform as well as max-norm minimization, and indeed, max-norm minimization is minimax-rate optimal in a sense made precise in the paper. While this behavior had been observed in numerical examples, this paper provides a mathematical justification for these observations.

There is a paper [28] in which it is claimed that the matrix completion problem can be solved using a Ramanujan graph to generate the sampling matrix. However, there is one step in the proof that does not appear to be justified. Thus the contents of Section 7.5.4 are apparently the first to provide a deterministic method for matrix completion.

Some of the content in this chapter is from the forthcoming Ph.D. theses of the author's students Shantanu Prasad Burnwal (see Section 7.5.4) and Shashank Ranjan (see Section 7.2.3).