Contents

List of Figures **xvii**

Preface **xix**
1 Objectives and Scope of the Book . . . xix
2 Overview of the Second Edition . . . xx
3 Intended Audience . . . xxii
4 Acknowledgments . . . xxiii

1 Introduction: Examples, Background, and Perspectives **1**
1 Orientation . . . 1
1.1 Geometry as a Variable . . . 1
1.2 Outline of the Introductory Chapter . . . 3
2 A Simple One-Dimensional Example . . . 3
3 Buckling of Columns . . . 4
4 Eigenvalue Problems . . . 6
5 Optimal Triangular Meshing . . . 7
6 Modeling Free Boundary Problems . . . 10
6.1 Free Interface between Two Materials . . . 11
6.2 Minimal Surfaces . . . 12
7 Design of a Thermal Diffuser . . . 13
7.1 Description of the Physical Problem . . . 13
7.2 Statement of the Problem . . . 14
7.3 Reformulation of the Problem . . . 16
7.4 Scaling of the Problem . . . 16
7.5 Design Problem . . . 17
8 Design of a Thermal Radiator . . . 18
8.1 Statement of the Problem . . . 18
8.2 Scaling of the Problem . . . 20
9 A Glimpse into Segmentation of Images . . . 21
9.1 Automatic Image Processing . . . 21
9.2 Image Smoothing/Filtering by Convolution and Edge Detectors 22
9.2.1 Construction of the Convolution of I . . . 23
9.2.2 Space-Frequency Uncertainty Relationship . . . 23
9.2.3 Laplacian Detector . . . 25

9.3 Objective Functions Defined on the Whole Edge 26
9.3.1 Eulerian Shape Semiderivative 26
9.3.2 From Local to Global Conditions on the Edge . . . 27
9.4 Snakes, Geodesic Active Contours, and Level Sets 28
9.4.1 Objective Functions Defined on the Contours 28
9.4.2 Snakes and Geodesic Active Contours 28
9.4.3 Level Set Method 29
9.4.4 Velocity Carried by the Normal 30
9.4.5 Extension of the Level Set Equations 31
9.5 Objective Function Defined on the Whole Image 32
9.5.1 Tikhonov Regularization/Smoothing 32
9.5.2 Objective Function of Mumford and Shah 32
9.5.3 Relaxation of the $(N-1)$-Hausdorff Measure 33
9.5.4 Relaxation to BV-, H^s-, and SBV-Functions 33
9.5.5 Cracked Sets and Density Perimeter 35
10 Shapes and Geometries: Background and Perspectives 36
10.1 Parametrize Geometries by Functions or Functions by Geometries? 36
10.2 Shape Analysis in Mechanics and Mathematics 39
10.3 Characteristic Functions: Surface Measure and Geometric Measure Theory 41
10.4 Distance Functions: Smoothness, Normal, and Curvatures . . 41
10.5 Shape Optimization: Compliance Analysis and Sensitivity Analysis 43
10.6 Shape Derivatives 44
10.7 Shape Calculus and Tangential Differential Calculus 46
10.8 Shape Analysis in This Book 46
11 Shapes and Geometries: Second Edition 47
11.1 Geometries Parametrized by Functions 48
11.2 Functions Parametrized by Geometries 50
11.3 Shape Continuity and Optimization 52
11.4 Derivatives, Shape and Tangential Differential Calculuses, and Derivatives under State Constraints 53

2 Classical Descriptions of Geometries and Their Properties 55
1 Introduction 55
2 Notation and Definitions 56
2.1 Basic Notation 56
2.2 Abelian Group Structures on Subsets of a Fixed Holdall D . 56
2.2.1 First Abelian Group Structure on $(\mathcal{P}(D), \triangle)$ 57
2.2.2 Second Abelian Group Structure on $(\mathcal{P}(D), \triangledown)$ . . . 58
2.3 Connected Space, Path-Connected Space, and Geodesic Distance 58
2.4 Bouligand's Contingent Cone, Dual Cone, and Normal Cone 59
2.5 Sobolev Spaces 60
2.5.1 Definitions 60

2.5.2 The Space $W_0^{m,p}(\Omega)$ . . . 61
2.5.3 Embedding of $H_0^1(\Omega)$ into $H_0^1(D)$ . . . 62
2.5.4 Projection Operator . . . 63
2.6 Spaces of Continuous and Differentiable Functions . . . 63
2.6.1 Continuous and C^k Functions . . . 63
2.6.2 Hölder ($C^{0,\ell}$) and Lipschitz ($C^{0,1}$) Continuous Functions . . . 65
2.6.3 Embedding Theorem . . . 65
2.6.4 Identity $C^{k,1}(\overline{\Omega}) = W^{k+1,\infty}(\Omega)$: From Convex to Path-Connected Domains via the Geodesic Distance 66
3 Sets Locally Described by an Homeomorphism or a Diffeomorphism 67
3.1 Sets of Classes C^k and $C^{k,\ell}$ . . . 67
3.2 Boundary Integral, Canonical Density, and Hausdorff Measures 70
3.2.1 Boundary Integral for Sets of Class C^1 . . . 70
3.2.2 Integral on Submanifolds . . . 71
3.2.3 Hausdorff Measures . . . 72
3.3 Fundamental Forms and Principal Curvatures . . . 73
4 Sets Globally Described by the Level Sets of a Function . . . 75
5 Sets Locally Described by the Epigraph of a Function . . . 78
5.1 Local C^0 Epigraphs, C^0 Epigraphs, and Equi-C^0 Epigraphs and the Space $\mathcal{H}$ of Dominating Functions . . . 79
5.2 Local $C^{k,\ell}$-Epigraphs and Hölderian/Lipschitzian Sets . . . 87
5.3 Local $C^{k,\ell}$-Epigraphs and Sets of Class $C^{k,\ell}$ . . . 89
5.4 Locally Lipschitzian Sets: Some Examples and Properties . . 92
5.4.1 Examples and Continuous Linear Extensions . . . 92
5.4.2 Convex Sets . . . 93
5.4.3 Boundary Measure and Integral for Lipschitzian Sets 94
5.4.4 Geodesic Distance in a Domain and in Its Boundary 97
5.4.5 Nonhomogeneous Neumann and Dirichlet Problems 100
6 Sets Locally Described by a Geometric Property . . . 101
6.1 Definitions and Main Results . . . 102
6.2 Equivalence of Geometric Segment and C^0 Epigraph Properties . . . 104
6.3 Equivalence of the Uniform Fat Segment and the Equi-C^0 Epigraph Properties . . . 109
6.4 Uniform Cone/Cusp Properties and Hölderian/Lipschitzian Sets . . . 113
6.4.1 Uniform Cone Property and Lipschitzian Sets . . . 114
6.4.2 Uniform Cusp Property and Hölderian Sets . . . 115
6.5 Hausdorff Measure and Dimension of the Boundary . . . 116

3 Courant Metrics on Images of a Set 123
1 Introduction . . . 123
2 Generic Constructions of Micheletti . . . 124
2.1 Space $\mathcal{F}(\Theta)$ of Transformations of $\mathbf{R}^N$ . . . 124
2.2 Diffeomorphisms for $\mathcal{B}(\mathbf{R}^N, \mathbf{R}^N)$ and $C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$ . . . 136

2.3 Closed Subgroups $\mathcal{G}$ . . . 138
2.4 Courant Metric on the Quotient Group $\mathcal{F}(\Theta)/\mathcal{G}$ . . . 140
2.5 Assumptions for $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$, $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, and $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$ 143
2.5.1 Checking the Assumptions . . . 143
2.5.2 Perturbations of the Identity and Tangent Space . . 147
2.6 Assumptions for $C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ and $C_0^{k,1}(\mathbf{R}^N, \mathbf{R}^N)$ . . . 149
2.6.1 Checking the Assumptions . . . 149
2.6.2 Perturbations of the Identity and Tangent Space . . 151
3 Generalization to All Homeomorphisms and C^k-Diffeomorphisms . . 153

4 Transformations Generated by Velocities 159
1 Introduction . . . 159
2 Metrics on Transformations Generated by Velocities . . . 161
2.1 Subgroup G_Θ of Transformations Generated by Velocities . . 161
2.2 Complete Metrics on G_Θ and Geodesics . . . 166
2.3 Constructions of Azencott and Trouvé . . . 169
3 Semiderivatives via Transformations Generated by Velocities . . . 170
3.1 Shape Function . . . 170
3.2 Gateaux and Hadamard Semiderivatives . . . 170
3.3 Examples of Families of Transformations of Domains . . . 173
3.3.1 C^∞-Domains . . . 173
3.3.2 C^k-Domains . . . 175
3.3.3 Cartesian Graphs . . . 176
3.3.4 Polar Coordinates and Star-Shaped Domains . . . 177
3.3.5 Level Sets . . . 178
4 Unconstrained Families of Domains . . . 180
4.1 Equivalence between Velocities and Transformations . . . 180
4.2 Perturbations of the Identity . . . 183
4.3 Equivalence for Special Families of Velocities . . . 185
5 Constrained Families of Domains . . . 193
5.1 Equivalence between Velocities and Transformations . . . 193
5.2 Transformation of Condition ($V2_D$) into a Linear Constraint . . . 200
6 Continuity of Shape Functions along Velocity Flows . . . 202

5 Metrics via Characteristic Functions 209
1 Introduction . . . 209
2 Abelian Group Structure on Measurable Characteristic Functions . . 210
2.1 Group Structure on $X_\mu(\mathbf{R}^N)$ . . . 210
2.2 Measure Spaces . . . 211
2.3 Complete Metric for Characteristic Functions in L^p-Topologies . . . 212
3 Lebesgue Measurable Characteristic Functions . . . 214
3.1 Strong Topologies and C^∞-Approximations . . . 214
3.2 Weak Topologies and Microstructures . . . 215
3.3 Nice or Measure Theoretic Representative . . . 220

3.4 The Family of Convex Sets . . . 223
3.5 Sobolev Spaces for Measurable Domains . . . 224
4 Some Compliance Problems with Two Materials . . . 228
4.1 Transmission Problem and Compliance . . . 228
4.2 The Original Problem of Céa and Malanowski . . . 235
4.3 Relaxation and Homogenization . . . 239
5 Buckling of Columns . . . 240
6 Caccioppoli or Finite Perimeter Sets . . . 244
6.1 Finite Perimeter Sets . . . 245
6.2 Decomposition of the Integral along Level Sets . . . 251
6.3 Domains of Class $W^{\varepsilon,p}(D)$, $0 \le \varepsilon < 1/p$, $p \ge 1$, and a Cascade of Complete Metric Spaces . . . 252
6.4 Compactness and Uniform Cone Property . . . 254
7 Existence for the Bernoulli Free Boundary Problem . . . 258
7.1 An Example: Elementary Modeling of the Water Wave . . . 258
7.2 Existence for a Class of Free Boundary Problems . . . 260
7.3 Weak Solutions of Some Generic Free Boundary Problems . . . 262
7.3.1 Problem without Constraint . . . 262
7.3.2 Constraint on the Measure of the Domain Ω . . . 264
7.4 Weak Existence with Surface Tension . . . 265

6 Metrics via Distance Functions 267
1 Introduction . . . 267
2 Uniform Metric Topologies . . . 268
2.1 Family of Distance Functions $C_d(D)$ . . . 268
2.2 Pompéiu–Hausdorff Metric on $C_d(D)$ . . . 269
2.3 Uniform Complementary Metric Topology and $C_d^c(D)$ . . . 275
2.4 Families $C_d^c(E;D)$ and $C_{d,\mathrm{loc}}^c(E;D)$ . . . 278
3 Projection, Skeleton, Crack, and Differentiability . . . 279
4 $W^{1,p}$-Metric Topology and Characteristic Functions . . . 292
4.1 Motivations and Main Properties . . . 292
4.2 Weak $W^{1,p}$-Topology . . . 296
5 Sets of Bounded and Locally Bounded Curvature . . . 299
5.1 Examples . . . 301
6 Reach and Federer's Sets of Positive Reach . . . 303
6.1 Definitions and Main Properties . . . 303
6.2 C^k-Submanifolds . . . 310
6.3 A Compact Family of Sets with Uniform Positive Reach . . . 315
7 Approximation by Dilated Sets/Tubular Neighborhoods and Critical Points . . . 316
8 Characterization of Convex Sets . . . 318
8.1 Convex Sets and Properties of d_A . . . 318
8.2 Semiconvexity and BV Character of d_A . . . 320
8.3 Closed Convex Hull of A and Fenchel Transform of d_A . . . 322
8.4 Families of Convex Sets $\mathcal{C}_d(D)$, $\mathcal{C}_d^c(D)$, $\mathcal{C}_d^c(E;D)$, and $\mathcal{C}_{d,\mathrm{loc}}^c(E;D)$ . . . 323

9 Compactness Theorems for Sets of Bounded Curvature 324
9.1 Global Conditions in D 325
9.2 Local Conditions in Tubular Neighborhoods 327

7 Metrics via Oriented Distance Functions 335
1 Introduction 335
2 Uniform Metric Topology 337
2.1 The Family of Oriented Distance Functions $C_b(D)$ 337
2.2 Uniform Metric Topology 339
3 Projection, Skeleton, Crack, and Differentiability 344
4 $W^{1,p}(D)$-Metric Topology and the Family $C_b^0(D)$ 349
4.1 Motivations and Main Properties 349
4.2 Weak $W^{1,p}$-Topology 352
5 Boundary of Bounded and Locally Bounded Curvature 354
5.1 Examples and Limit of Tubular Norms as h Goes to Zero . . 355
6 Approximation by Dilated Sets/Tubular Neighborhoods 358
7 Federer's Sets of Positive Reach 361
7.1 Approximation by Dilated Sets/Tubular Neighborhoods . . . 361
7.2 Boundaries with Positive Reach 363
8 Boundary Smoothness and Smoothness of b_A 365
9 Sobolev or $W^{m,p}$ Domains 373
10 Characterization of Convex and Semiconvex Sets 375
10.1 Convex Sets and Convexity of $b_{\overline{A}}$ 375
10.2 Families of Convex Sets $\mathcal{C}_b(D)$, $\mathcal{C}_b(E;D)$, and $\mathcal{C}_{b,\mathrm{loc}}(E;D)$ 379
10.3 BV Character of b_A and Semiconvex Sets 380
11 Compactness and Sets of Bounded Curvature 381
11.1 Global Conditions on D 382
11.2 Local Conditions in Tubular Neighborhoods 382
12 Finite Density Perimeter and Compactness 385
13 Compactness and Uniform Fat Segment Property 387
13.1 Main Theorem 387
13.2 Equivalent Conditions on the Local Graph Functions 391
14 Compactness under the Uniform Fat Segment Property and a Bound on a Perimeter 393
14.1 De Giorgi Perimeter of Caccioppoli Sets 393
14.2 Finite Density Perimeter 394
15 The Families of Cracked Sets 394
16 A Variation of the Image Segmentation Problem of Mumford and Shah 400
16.1 Problem Formulation 400
16.2 Cracked Sets without the Perimeter 401
16.2.1 Technical Lemmas 401
16.2.2 Another Compactness Theorem 402
16.2.3 Proof of Theorem 16.1 402
16.3 Existence of a Cracked Set with Minimum Density Perimeter 405

16.4 Uniform Bound or Penalization Term in the Objective Function on the Density Perimeter . . . 407

8 Shape Continuity and Optimization 409
1 Introduction and Generic Examples . . . 409
1.1 First Generic Example . . . 411
1.2 Second Generic Example . . . 411
1.3 Third Generic Example . . . 411
1.4 Fourth Generic Example . . . 412
2 Upper Semicontinuity and Maximization of the First Eigenvalue . . . 412
3 Continuity of the Transmission Problem . . . 417
4 Continuity of the Homogeneous Dirichlet Boundary Value Problem . . . 418
4.1 Classical, Relaxed, and Overrelaxed Problems . . . 418
4.2 Classical Dirichlet Boundary Value Problem . . . 421
4.3 Overrelaxed Dirichlet Boundary Value Problem . . . 423
4.3.1 Approximation by Transmission Problems . . . 423
4.3.2 Continuity with Respect to $\mathrm{X}(D)$ in the $L^p(D)$-Topology . . . 424
4.4 Relaxed Dirichlet Boundary Value Problem . . . 425
5 Continuity of the Homogeneous Neumann Boundary Value Problem 426
6 Elements of Capacity Theory . . . 429
6.1 Definition and Basic Properties . . . 429
6.2 Quasi-continuous Representative and H^1-Functions . . . 431
6.3 Transport of Sets of Zero Capacity . . . 432
7 Crack-Free Sets and Some Applications . . . 434
7.1 Definitions and Properties . . . 434
7.2 Continuity and Optimization over $L(D, r, \mathcal{O}, \lambda)$ . . . 437
7.2.1 Continuity of the Classical Homogeneous Dirichlet Boundary Condition . . . 437
7.2.2 Minimization/Maximization of the First Eigenvalue . . . 438
8 Continuity under Capacity Constraints . . . 440
9 Compact Families $\mathcal{O}_{c,r}(D)$ and $L_{c,r}(\mathcal{O}, D)$ . . . 447
9.1 Compact Family $\mathcal{O}_{c,r}(D)$ . . . 447
9.2 Compact Family $L_{c,r}(\mathcal{O}, D)$ and Thick Set Property . . . 450
9.3 Maximizing the Eigenvalue $\lambda^A(\Omega)$ . . . 452
9.4 State Constrained Minimization Problems . . . 453
9.5 Examples with a Constraint on the Gradient . . . 454

9 Shape and Tangential Differential Calculuses 457
1 Introduction . . . 457
2 Review of Differentiation in Topological Vector Spaces . . . 458
2.1 Definitions of Semiderivatives and Derivatives . . . 458
2.2 Derivatives in Normed Vector Spaces . . . 461
2.3 Locally Lipschitz Functions . . . 465
2.4 Chain Rule for Semiderivatives . . . 465

2.5 Semiderivatives of Convex Functions . . . 467
2.6 Hadamard Semiderivative and Velocity Method . . . 469
3 First-Order Shape Semiderivatives and Derivatives . . . 471
3.1 Eulerian and Hadamard Semiderivatives . . . 471
3.2 Hadamard Semidifferentiability and Courant Metric Continuity . . . 476
3.3 Perturbations of the Identity and Gateaux and Fréchet Derivatives . . . 476
3.4 Shape Gradient and Structure Theorem . . . 479
4 Elements of Shape Calculus . . . 482
4.1 Basic Formula for Domain Integrals . . . 482
4.2 Basic Formula for Boundary Integrals . . . 484
4.3 Examples of Shape Derivatives . . . 486
4.3.1 Volume of Ω and Surface Area of Γ . . . 486
4.3.2 $H^1(\Omega)$-Norm . . . 487
4.3.3 Normal Derivative . . . 488
5 Elements of Tangential Calculus . . . 491
5.1 Intrinsic Definition of the Tangential Gradient . . . 492
5.2 First-Order Derivatives . . . 495
5.3 Second-Order Derivatives . . . 496
5.4 A Few Useful Formulae and the Chain Rule . . . 497
5.5 The Stokes and Green Formulae . . . 498
5.6 Relation between Tangential and Covariant Derivatives . . . 498
5.7 Back to the Example of Section 4.3.3 . . . 501
6 Second-Order Semiderivative and Shape Hessian . . . 501
6.1 Second-Order Derivative of the Domain Integral . . . 502
6.2 Basic Formula for Domain Integrals . . . 504
6.3 Nonautonomous Case . . . 505
6.4 Autonomous Case . . . 510
6.5 Decomposition of $d^2J(\Omega; V(0), W(0))$ . . . 515

10 Shape Gradients under a State Equation Constraint 519
1 Introduction . . . 519
2 Min Formulation . . . 521
2.1 An Illustrative Example and a Shape Variational Principle . . . 521
2.2 Function Space Parametrization . . . 522
2.3 Differentiability of a Minimum with Respect to a Parameter . . . 523
2.4 Application of the Theorem . . . 526
2.5 Domain and Boundary Integral Expressions of the Shape Gradient . . . 530
3 Buckling of Columns . . . 532
4 Eigenvalue Problems . . . 535
4.1 Transport of $H_0^k(\Omega)$ by $W^{k,\infty}$-Transformations of $\mathbf{R}^N$ . . . 536
4.2 Laplacian and Bi-Laplacian . . . 537
4.3 Linear Elasticity . . . 546

5 Saddle Point Formulation and Function Space Parametrization . . . 551
5.1 An Illustrative Example . . . 551
5.2 Saddle Point Formulation . . . 552
5.3 Function Space Parametrization . . . 553
5.4 Differentiability of a Saddle Point with Respect to a Parameter . . . 555
5.5 Application of the Theorem . . . 559
5.6 Domain and Boundary Expressions for the Shape Gradient . 561
6 Multipliers and Function Space Embedding . . . 562
6.1 The Nonhomogeneous Dirichlet Problem . . . 562
6.2 A Saddle Point Formulation of the State Equation . . . 563
6.3 Saddle Point Expression of the Objective Function . . . 564
6.4 Verification of the Assumptions of Theorem 5.1 . . . 566

Elements of Bibliography **571**

Index of Notation **615**

Index **619**