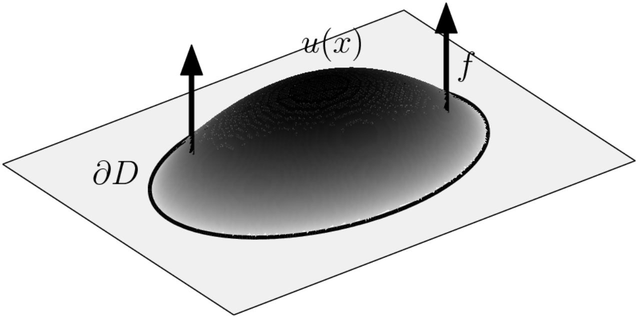
Elastic Membrane



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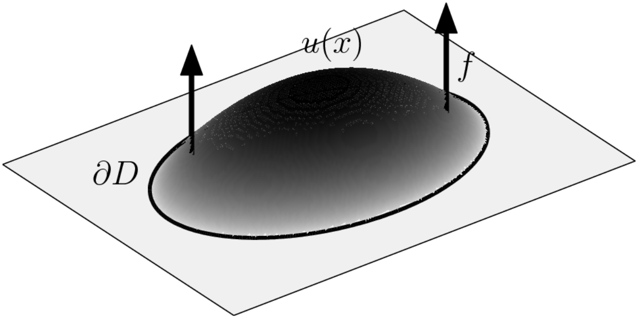

vertical force with density $f(x_1, x_2) \to$ displacement $u(x_1, x_2)$

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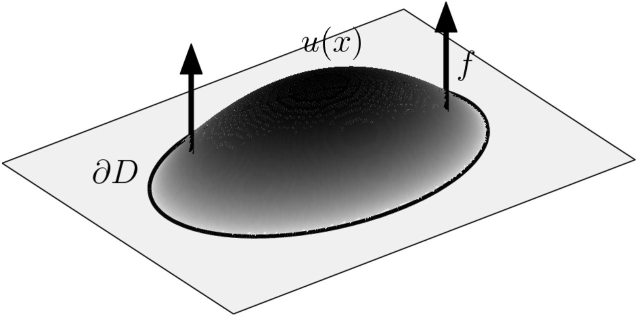

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Poisson equation

$$-\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D$$

Poisson Problem

classical solution: twice continuously differentiable in D, continuous on $\bar{D}$

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$$\int_D \text{grad}\, u \text{ grad}\, v = \int_D fv, \quad \forall v \in H_0^1(D)$$

$\Leftrightarrow$

$$\mathcal{Q}(u) = \min_{v \in H_0^1(D)} \mathcal{Q}(v), \quad \mathcal{Q}(v) = \frac{1}{2} \int_D |\text{grad}\, v|^2 - \int_D fv$$

Proof

(i) u classical solution, $v_{|\partial D} = 0$
integrate differential equation by parts $\rightsquigarrow$ variational equations

$$\int_D fv = -\int_D \Delta u \, v = \int_D \operatorname{grad} u \operatorname{grad} v$$

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(ii) characterization of a solution u of the minimization problem

$$\mathcal{Q}(u + tv) \geq \mathcal{Q}(u) = \frac{1}{2} \int_D |\operatorname{grad} u|^2 - \int_D fu$$

simplifications $\rightsquigarrow$

$$t \left[\int_D \operatorname{grad} u \operatorname{grad} v - \int_D fv \right] + \frac{1}{2} t^2 \int_D |\operatorname{grad} v|^2 \geq 0$$

$t \in \mathbb{R}$ arbitrary: $[\ldots] = 0 \Leftrightarrow$ variational equations

simplification details
left side

$$
\begin{aligned}
\mathcal{Q}(u+tv) &= \frac{1}{2}\int_D \operatorname{grad}(u+tv)\operatorname{grad}(u+tv) - \int_D f\,(u+tv) \\
&= \left\{\frac{1}{2}\int_D |\operatorname{grad} u|^2 - \int_D f\,u\right\} + \\
&\qquad t\left[\int_D \operatorname{grad} u \operatorname{grad} v - \int_D f\,v\right] + \frac{1}{2}t^2\int_D |\operatorname{grad} v|^2
\end{aligned}
$$

$\{\ldots\} = \mathcal{Q}(u)$

Ritz–Galerkin Approximation of Poisson's Problem

The coefficients of a standard finite element approximation

$$u_h = \sum_i u_i B_i, \quad B_{i|\partial D} = 0$$

for the boundary value problem

$$-\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D\,,$$

are determined from the linear system $GU = F$ with

$$g_{k,i} = \int_D \operatorname{grad} B_i \operatorname{grad} B_k, \quad f_k = \int_D f\, B_k\,.$$

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(i) variational equations $\int \operatorname{grad} u \operatorname{grad} v = \int fv$

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$u = u_h = \sum_i u_i B_i,\ v = B_k \rightsquigarrow$

$$\int_D \text{grad} \left(\sum_i u_i B_i \right) \text{grad}\, B_k = \int_D f\, B_k$$

Proof

(i) variational equations $\int \operatorname{grad} u \operatorname{grad} v = \int fv$
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linear system

$$\sum_i g_{k,i} u_i = f_k \quad \Leftrightarrow \quad GU = F$$

(ii) minimization of $\mathcal{Q}(u) = \frac{1}{2}\int |\operatorname{grad} u|^2 - \int fu$

$u = u_h \rightsquigarrow$

$$\frac{1}{2}\int_D \operatorname{grad}\left(\sum_i u_i B_i\right) \operatorname{grad}\left(\sum_k u_k B_k\right) - \int_D f \sum_k u_k B_k \to \min$$

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quadratic form

$$\frac{1}{2}\sum_{i,k} u_k g_{k,i} u_i - \sum_k f_k u_k \quad \Leftrightarrow \quad \frac{1}{2} UGU - FU$$

with symmetric, positive definite matrix G

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minimized by $U \Leftrightarrow GU = F$