Abstract Variational Problem

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This weak form of the boundary value problem is well suited for numerical approximations, in particular because it requires less regularity. For a differential operator of order $2m$, the existence of weak derivatives up to order m suffices.

Ritz–Galerkin Approximation

The Ritz–Galerkin approximation $u_h = \sum_i u_i B_i \in \mathbb{B}_h \subset H$ of the variational problem

$$a(u, v) = \lambda(v), \quad v \in H,$$

is determined by the linear system

$$\sum_i a(B_i, B_k)\, u_i = \lambda(B_k),$$

which we abbreviate as $GU = F$.

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Hilbert space: $H = H_0^1(D)$
simple finite element subspace $\mathbb{B}_h$:
piecewise linear functions on a triangulation of D

Ellipticity

A bilinear form a on a Hilbert space H is elliptic if it is bounded and equivalent to the norm on H, i.e., if for all $u, v \in H$

$$|a(u, v)| \leq c_b \|u\| \|v\|, \quad c_e \|u\|^2 \leq a(u, u)$$

with positive constants c_b and c_e.

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Cauchy–Schwarz inequality $\implies$

$$\begin{aligned} |a(u,v)| &\leq a(u,u)^{1/2}\, a(v,v)^{1/2} = \left(\int_D \|\operatorname{grad} u\|^2\right)^{1/2} \left(\int_D \|\operatorname{grad} v\|^2\right)^{1/2} \\ &\leq \|u\|_1 \|v\|_1 \end{aligned}$$

where

$$\|w\|_1 = \left(\int_D |w|^2 + \|\operatorname{grad} w\|^2\right)^{1/2}$$

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add $\int_D \|\text{grad}\, u\|^2 = a(u,u)$ to both sides $\rightsquigarrow$

$$\|u\|_1^2 \leq (\text{const}(D)+1) \int_D \|\text{grad}\, u\|^2\,,$$

i.e., $c_e = (\text{const}(D)+1)^{-1}$

Lax–Milgram Theorem

If a is an elliptic bilinear form and λ is a bounded linear functional on a Hilbert space H, then the variational problem

$$a(u, v) = \lambda(v), \quad v \in V,$$

has a unique solution $u \in V$ for any closed subspace V of H. Moreover, if a is symmetric, the solution u can be characterized as the minimum of the quadratic form

$$\mathcal{Q}(u) = \frac{1}{2} a(u, u) - \lambda(u)$$

on V.

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existence of $G^{-1} \implies$ unique solvability of the Ritz–Galerkin system
$\rightsquigarrow$ part of the Lax–Milgram theorem, relevant for numerical schemes

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Riesz theorem $\rightsquigarrow$ representation for bounded linear functionals $\varrho : V \to \mathbb{R}$:

$$\varrho(v) = \langle \mathcal{R}\varrho, v \rangle$$

with $\langle \cdot, \cdot \rangle$ the scalar product on V (identical to the scalar product on H)
and $\mathcal{R}$ an isometry onto V

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for $\omega = c_e/c_b^2$

$$\langle \ldots, \ldots \rangle = \|u\|^2 - 2\omega a(u,u) + \omega^2\|\mathcal{R}\mathcal{A}u\|^2 \leq 1 - 2\omega c_e + \omega^2 c_b^2 < 1$$

since $\langle \mathcal{R}\mathcal{A}u, u\rangle = (\mathcal{A}u)(u) = a(u,u)$

Proof (symmetric case)

symmetric elliptic bilinear form $a \rightsquigarrow$ scalar product and equivalent norm $\|\cdot\|_a$:

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minimizing $\mathcal{Q} \Leftrightarrow$ best approximation to $R\lambda$ from V
characterization of the best approximation u,

$$a(u - R\lambda, v) = 0, \quad v \in V,$$

$\Leftrightarrow$ variational equations