Finite Element Methods with B-Splines

Problem Collection

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Contents

Part I

Problems

1 Introduction

Problem 1.0.1 Program: Poisson Problem on a Disc
Write a program [u,error] = error_poisson_radial(H,n) which solves the radially symmetric boundary value problem

$$-\Delta u = 1 \text{ in } D, \quad u = 0 \text{ on } \partial D\,,$$

with B-splines of degree n and grid width $1/H$ on an annulus D with radii $1/4$ and $1/2$. It returns the values of the computed solution at the centers of the grid cells and, by comparing with the exact solution, the error in the maximum norm.

Answer
error for $n = 3$ and $H = 10$: [ ]

2 Basic Finite Element Concepts

2.1 Model Problem

Problem 2.1.1 Boundary Value Problem for a Univariate Energy Functional
Characterize a smooth solution u of the minimization problem

$$u(1)^2 + \int_0^1 xu'(x)^2 + u(x)\,dx \quad \to \quad \min$$

in terms of a differential equation with boundary condition and determine u explicitly.
Answer
$u(1/2)$: ☐

Problem 2.1.2 Minimization of a Univariate Energy Functional
Solve the minimization problem

$$\min_{u(0)=0} \int_0^1 (u')^2 + 4u^2 - 3u\,.$$

Answer
$u(1)$: ☐

Problem 2.1.3 Energy Functional for a Univariate Boundary Value Problem
Solve the boundary value problem

$$-\frac{d}{dx}\left(\exp(x)\,\frac{d}{dx}u(x)\right) = 1, \quad u(0) = u(1) = 0\,,$$

and characterize the solution in terms of a minimization problem.
Answer
$u(1/2)$: ☐

Problem 2.1.4 Ritz-Galerkin Approximation with Sine Functions
Determine the Ritz-Galerkin approximation of the boundary value problem

$$-u'' + u = x, \quad u(0) = u(\pi) = 0\,,$$

for the finite elements $x \mapsto \sin(kx)$, $k = 1, \ldots, n$.
Answer
coefficient of $\sin(10\,x)$: ☐

Problem 2.1.5 Ritz-Galerkin System for a Univariate Energy Functional
Determine the entries of the matrix and the right hand side of the Ritz-Galerkin system for the minimization problem

$$\min_{u(0)=0} \int_0^1 (u'(x))^2 + xu(x)^2 - x^2u(x)\,dx$$

using the basis functions $B_k(x) = x^k$, $k = 1, \ldots, n$.
Answer
third diagonal element of the matrix: ☐

Problem 2.1.6 Ritz-Galerkin System for Hat-Functions of Univariate Variational Equations
Determine the entries of the matrix and the right side of the Ritz-Galerkin system for hat-functions b_i, $i = 1, \ldots, 1/h - 1$, with support $[ih - h, ih + h]$ for the variational equations

$$\int_0^1 (1+x)u'(x)v'(x) + u(x)v(x)\,dx = \int_0^1 v(x)\,dx, \quad v \in H_0^1(0,1)\,.$$

Answer
second diagonal element of the matrix for $h = 1/10$: ☐

Problem 2.1.7 Program: Ritz-Galerkin Approximation of a Radially Symmetric Poisson problem
Write a program `residuum = residuum_poisson_radial(n)` which computes and plots the Ritz-Galerkin approximation u_n of the radially symmetric Poisson problem

$$-\frac{1}{r}(ru')' = \exp(r^2), \quad u(1) = 0\,,$$

on the unit disc for the basis functions

$$B_1, \ldots, B_n, \quad B_k(r) = 1 - r^{2k}\,.$$

Moreover, the program calculates the maximum norm of the residuum $e_n(r) = -(1/r)(ru_n')' - \exp(r^2)$.
Answer
maximum norm of the residuum for $n = 5$: ☐

2.2 Mesh Based Elements

Problem 2.2.1 Program: Finite Element Solution of a Univariate Boundary Value Problem with Hat-Functions
Write a program `[u,error] = minimization_problem(H)` which solves the minimization problem

$$\frac{1}{2}\int_0^1 (1+x)\,(u'(x))^2\,dx - \int_0^1 u(x)\,dx \to \min, \quad u(0) = u(1) = 0\,,$$

numerically using hat-functions with grid width $1/H$ as finite elements. Compute the error by comparing with the exact solution $u(x) = -x + \ln(1+x)/\ln 2$ at the grid points $x = kh$.
Answer
error for $H = 100$: ☐

Problem 2.2.2 Ritz-Galerkin Matrix for Hat-Functions
Compute the contributions to the Ritz-Galerkin matrix

$$G: \int \operatorname{grad} B_j \operatorname{grad} B_k$$

for the triangle with vertices

$$(0,0),\ (2,0),\ (1,3)$$

and the relevant hat-functions.
Answer
sum of the absolute values of the entries of the contributing 3×3-matrix: ☐

Problem 2.2.3 Finite Element Integrals for Hat-Functions on a Tetrahedron
Using the cubature formula

$$\int_\tau g \approx \operatorname{vol}\tau \left(-\frac{1}{20} \sum_m g(p_m) + \frac{1}{5} \sum_{m<m'} g((p_m + p_{m'})/2) \right)$$

for a simplex τ with vertices p_i, p_j, p_k, p_ℓ, derive a formula for the contributions $\int_\tau f B_m$, $m \in \{i, j, k, \ell\}$, to the right hand side of the Ritz-Galerkin system for hat-functions B_m and a linear function f.
Answer
largest weight of f_m for a simplex with volume 1: ☐

Problem 2.2.4 L_2-Norm of a Bivariate Hat-Function
Express the L_2-norm of a bivariate hat-function in terms of the areas of the triangles of its support. Which result do you obtain for equilateral triangles?
Answer
L_2-norm for a hat-function supported on a regular hexagon with side length 1: ☐

Problem 2.2.5 Program: Refinement of a Triangulation
Write a program `[t,p] = refine_triangulation(T,P)` which refines a triangulation by subdividing each triangle into 4 congruent subtriangles. Describe the triangulation by an array P of vertices and an array T which contains the vertex numbers of the triangles, i.e.,

$$P(T(k,1), 1:2),\ P(T(k,2), 1:2),\ P(T(k,3), 1:2)$$

are the vertices of the k-th triangle.
Answer
$\sum_{j,k} p_{j,k}^2$ after 3 refinements of the triangle with vertices $(0,1), (5,2), (3,4)$: ☐

Problem 2.2.6 Ritz-Galerkin Matrix for Hat-Functions on a Regular Triangulation
Which entries occur in a Ritz-Galerkin matrix for Poisson's problem and hat-functions on a triangulation consisting of equilateral triangles of side length h as finite elements?
Answer
sum of squares of the entries in a row: ☐

Problem 2.2.7 Dimension of Continuous Piecewise Polynomials on a Planar Triangulation
Determine the dimension $d(t, v, n)$ of the space of continuous piecewise polynomials of total degree $\leq n$ on a triangulation of a planar simply connected polygonal domain consisting of t triangles with v vertices. Which result do you obtain for a regular triangulation of a square consisting of $2m^2$ isosceles rectangular triangles?
Answer
$d(100, 100, 10)$: [ ]

Problem 2.2.8 Program: Evaluation of Lagrange Functions for Simplices
Write a program `L = lagrange_simplex(x,P,k)` which evaluates the Lagrange element associated with the node

$$(k_1(p_{1,1}, \ldots, p_{1,d}) + \cdots + k_{d+1}(p_{d+1,1}, \ldots, p_{d+1,d}))\,/n \qquad (\sum k_\nu = n)$$

of the simplex with vertices $(p_{\nu,1}, \ldots, p_{\nu,d}) \in \mathbb{R}^d$ at $(x_1, \ldots, x_d)$.
Answer
value at $x = (1, 2, 3)/8$ for the simplex spanned by the origin and the unit vectors and $k = (4, 5, 6, 7)$: [ ]

Problem 2.2.9 Bijectivity of a Bilinear Isoparametric Transformation of the Unit Square
Show that a bilinear isoparametric transformation of the unit square is bijective if and only if the image is a convex quadrilateral.

Problem 2.2.10 Program: L_2-Norm for Linear Combinations of Bivariate Hat-Functions
Write a program `s = norm_hat(f,p,t)` which computes the L_2-norm of a linear combination of bivariate hat-functions with values f_k at the vertices $(p_{k,1}, p_{k,2})$ of a triangulation specified by a list of the indices $(t_{j,1}, t_{j,2}, t_{j,3})$ of the triangle vertices.
Answer
L_2-norm for the test data listed below: [ ]

```
f=[1 2 1 2 3 1 1 0]
p=[0 0;1 0;2 0;0 1;1 1;0 2;2 2;1 3]
t=[1 2 5;1 4 5;2 3 5;3 5 7;4 5 6;5 6 8;5 7 8]
```

2.3 Sobolev Spaces

Problem 2.3.1 Sine Expansion for Poisson's Problem
Solve the Poisson problem

$$-\Delta u = f \text{ in } D = (0, 1)^2, \quad u = 0 \text{ on } \partial D$$

for $f(x) = x_1(1 - 2x_2)$ via sine expansion.
Answer
sine coefficient with largest absolute value: [ ]

Problem 2.3.2 Integrability for the Radial Laplace Operator
For which $\alpha \neq 0$ is the function

$$x \mapsto \Delta r^\alpha, \quad r = (x_1^2 + \cdots + x_m^2)^{1/2},$$

a) integrable

b) square integrable

on the m-dimensional unit ball $D:\ r < 1$.
Answer
integrable as well as square integrable for $m = 10$ if $\alpha >$ $\square$

Problem 2.3.3 Corner Singularity for Poisson's Equation
Show that the solution of

$$-\Delta u = 1 \text{ in } D = (0,1)^2, \quad u = 0 \text{ on } \partial D$$

has a continuous gradient, but is not twice continuously differentiable on $\overline{D}$.
Answer
rate of decay of sine coefficients: $O(|k|^{-s})$ with $s =$ $\square$

Problem 2.3.4 Weak Derivative of a Discontinuous Function
Show that

$$f(x) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

is discontinuous at $x = (0,0)$, yet has weak first order derivatives. Are these derivatives square integrable?
Answer
square integrable [`yes`/`no`]: $\square$

Problem 2.3.5 H_0^1 as Subspace of H^1
Show that, for a bounded domain D, $H_0^1(D)$ is a proper subspace of $H^1(D)$.

2.4 Abstract Variational Problems

Problem 2.4.1 Riesz Representation of a Univariate Functional
Determine the Riesz representation $v = \mathcal{R}\lambda$ of the functional $\lambda(u) = u(0)$ on the Hilbert space $H_0^1(-1,1)$ with respect to the scalar product $\langle u, v\rangle = \int_{-1}^1 u'(x)v'(x)\,dx$.
Answer
$v(0) =$ $\square$

Problem 2.4.2 Riesz Representation of the Integral of a Function over the Unit Disc
Determine the Riesz representation of the functional

$$u \mapsto \int_D u, \quad D:\ x_1^2 + x_2^2 < 1,$$

on $H_0^1(D)$ with respect to the scalar product $\langle f, g\rangle = \int_D \operatorname{grad} f\ \operatorname{grad} g$.
Answer
value of the representing function at the origin: $\square$

Problem 2.4.3 Ellipticity Constants of a Bivariate Bilinear Form
Determine ellipticity constants for the bilinear form

$$a(u,v) = \iint_D 2u_x v_x - u_x v_y - u_y v_x + 2u_y v_y$$

on the Hilbert space $H_0^1(D)$, for a bounded domain $D \subset \mathbb{R}^2$, with respect to the norm $|\cdot|_1$.

Problem 2.4.4 Ellipticity of a Bilinear Form with Variable Coefficients
Prove the ellipticity of the bilinear form

$$a(u,v) = \int_D (1 - x_1 x_2) \operatorname{grad} u(x) \operatorname{grad} v(x) \; + \; \exp(x_1 x_2) u(x) v(x) \, dx$$

on $H^1(D)$, $D:\, x_1^2 + x_2^2 < 1$.

Problem 2.4.5 Ritz-Galerkin Projection onto Linear Functions on a Triangle
Determine the Ritz-Galerkin projection onto linear functions $\{1, x, y\}$ for the bilinear form

$$a(u,v) = \int_D \operatorname{grad} u \operatorname{grad} v + uv, \quad u, v \in H^1(D)\,,$$

and the linear functional $\lambda(v) = \int_D xy\, v(x,y)\, dxdy$ with $D:\, x, y > 0,\, x + y < 1$.
Answer
coefficent of x: ☐

Problem 2.4.6 Fixed-Point Iteration for a Symmetric Positive Definite Linear System
Following the arguments in the proof of the Lax-Milgram theorem, describe a fixed-point iteration for solving a linear system $Au = f$ with a symmetric matrix A with positive eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_n$. Use a relaxation parameter to achieve the optimal convergence rate.
Answer
optimal relaxation parameter for $\lambda_n = 9\lambda_1$: ☐

Problem 2.4.7 Existence and Uniqueness of a Minimum of a Univariate Energy Functional
Show that the quadratic energy functional

$$\mathcal{Q}(u) = \int_0^1 u'(x)^2 + u(x) \ln x \, dx \; - \; u(1/2)$$

has a unique minimum $u \in H_0^1(0,1)$.

2.5 Approximation Error

Problem 2.5.1 H^1-Error of Univariate Hat-Functions
Derive the error estimate

$$|u - u_h|_1 \leq h|u|_2$$

for piecewise linear interpolants u_h of a function $u \in H^2(0,1)$.

3 B-Splines

3.1 The Concept of Splines

Problem 3.1.1 Dimension of Univariate Quadratic Splines
Determine the dimension of univariate quadratic splines on a partition $0 < h < 2h < \ldots < 1-h < 1$ of $(0,1)$.
Answer
dimension for $h = 1/10$: ☐

Problem 3.1.2 Smoothness Constraints for Hermite Data of Cubic Splines
For cubic splines p with knots $h\mathbb{Z}$, determine the constraints on the data $p_k = p(kh)$ and $p'_k = p'(kh)$ which imply continuity of the second derivative.
Answer
quotient of the coefficients of $p'(kh)$ and $p'((k+1)h)$: ☐

3.2 Definition and Basic Properties

Problem 3.2.1 Highest Derivative of a B-Spline
Determine the values $d^n = (d_0^n, \ldots, d_n^n)$ of the n-th derivative of the uniform B-spline b^n.
Answer
$\sum_{k=0}^{10} |d_k^{10}| =$ ☐

Problem 3.2.2 Truncated Power Representation of a Uniform B-Spline
Show that a B-spline can be expressed as linear combination of truncated powers:

$$b^n(x) = \frac{1}{n!} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x-k)_+^n$$

where z_+^n equals 0 for $z < 0$ and z^n for $z \geq 0$.

3.3 Recurrence Relation

Problem 3.3.1 Program: Values and Derivatives of a Uniform B-Spline
Write a program `[b,db] = b_spline(n,H)` which computes values and derivatives of a uniform B-spline of degree n at the points $0, 1/H, \ldots, n+1$.
Answer
sum of absolute values of the B-spline and its derivatives for $n = 3$, $H = 4$: ☐

3.4 Representation of Polynomials

Problem 3.4.1 Polynomial Segments of a Quadratic Cardinal Spline
Determine the polynomial segments of the 2-periodic cardinal spline $x \mapsto p(x) = \sum_{k\in\mathbb{Z}} (-1)^k b^2(x-k)$.
Answer
$\sum_{x=0}^{1} \sum_{k=0}^{2} |p^{(k)}(x)/k!|$: ☐

Problem 3.4.2 Program, B-Spline Representation of Polynomials
Write a program `a = marsden_polynomial(p)` which determines the B-spline coefficients $c_k = \sum_{j=0}^{n} a_{j+1} k^j$ of a polynomial $p(x) = \sum_{j=0}^{n} p_{j+1} x^j$.
Answer
$\sum_j a_j$ for $p = (1,2,3,4)$: ☐

Problem 3.4.3 B-Spline Coefficients of a Monomial
Show that

$$x^2 = \sum_{k\in\mathbb{Z}} c_k b^n(x-k), \quad c_k = (k+(n+1)/2)^2 - (n+1)/12$$

for $n > 1$.

Problem 3.4.4 B-Spline Coefficients of a Cubic Polynomial
Represent the polynomial $p(x) = x^3 - 3x^2 + 2x - 4$ as linear combination of the cubic B-splines $b^3_{k,h}$.

Answer
coefficient of $b^3_{0,1}$: ☐

Problem 3.4.5 Minimal Support of a B-Spline
Show that the B-spline b^n has minimal support, i.e., if b is a spline of degree $\le n$ with knots at the integers which vanishes outside of $[0,n]$ or $[1,n+1]$, then $b=0$.

3.5 Subdivision

Problem 3.5.1 Program: Subdivision of a Univariate Cardinal Spline
Write a program `c = subdivision(c,n)`, which implements a subdivision step for a cardinal spline $\sum_k c_k b^n_{k,h}$.

Answer
$\sum_k c_k$ after subdivision for $c = (0,1,4,\ldots,9^2)$ and $n=3$: ☐

Problem 3.5.2 Convergence of B-Spline Coefficients for Subdivision
For a bounded spline $p = \sum_{k\in\mathbb{Z}} c_k b^n_{k,h}$ denote by $c_k = c^0_k, c^1_k, c^2_k, \ldots$, the B-spline coefficients generated by subdivision at midpoints. Show that

$$\left|c^\ell_k - p(x)\right| \le \gamma\, 2^{-\ell}, \quad x \in \operatorname{supp} b^n_{k,h^\ell},\ h_\ell = h2^{-\ell},$$

where the constant γ depends on n and $\max_k |c_k|$.

Problem 3.5.3 Subdivision of a Cardinal Spline
Describe an algorithm for subdividing a cardinal spline $\sum_k c_k b^n_{k,h}$ by splitting each grid interval $kh + [0,h]$ into m equal subintervals.

Answer
sum of refined coefficients c for $n=2$, $m=3$, and $c = (\ldots,0,1,0,\ldots)$: ☐

3.6 Scalar Products

Problem 3.6.1 Identities for Scalar Products of B-Splines and Their Derivatives
Show the following identities for the scalar products of univariate B-splines $b^n_{k,h}$ and of their derivatives:

a) $\sum_k s^n_k = h$

b) $\sum_k d^n_k = 0$

Problem 3.6.2 Program: Scalar Products of Univariate B-splines
Write a program `[s,d] = scalar_products(n)` which computes the scalar products $s^n_0,\ldots,s^n_{n-1}$ and $d^n_0,\ldots,d^n_{n-1}$ of univariate B-splines with grid width $h=1$ and their derivatives.

Answer
$\sum_k |s^{10}_k| + \sum_k |d^{10}_k|$: ☐

4 Finite Element Bases

4.1 Multivariate B-Splines

Problem 4.1.1 Values of Bivariate B-Splines at Grid Cell Midpoints
Determine the values at the midpoints $(x_1, x_2)/2$, $x_\nu \in \mathbb{Z}$, of the grid cells for the tensor product B-splines $b^n_{(0,0),1}$ of degree

a) $n = (1, 1)$,

b) $n = (2, 1)$,

c) $n = (2, 2)$.

Answer
sum of the different nonzero values in all cases: [ ]

Problem 4.1.2 Values and Gradients of a Bicubic B-Spline
Determine the values and gradients of the B-spline $b^{(3,3)}_{(0,0),1}$ at the grid of points with integer coordinates.
Answer
maximal value: [ ]

Problem 4.1.3 Values and Gradients of a Biquadratic B-Spline
Compute the values and the gradients of the B-spline $b^{(2,2)}_{(0,0),1}$ at the centers $\ell + (1/2, 1/2)$, $0 \le \ell_\nu \le 2$, of its support.
Answer
largest value: [ ]

Problem 4.1.4 Ritz-Galerkin Integrals of Bilinear B-Splines
Determine the Ritz-Galerkin integrals

$$\int \operatorname{grad} b^n_{k,h} \, \operatorname{grad} b^n_{\ell,h}$$

for bivariate tensor product B-splines of degree $n = (1, 1)$.
Answer
sum of absolute values of different integrals: [ ]

Problem 4.1.5 Laplace Operator, Applied to a Bicubic B-Spline
For the bicubic B-spline with support $[0, 4]^2$ compute $\Delta b^3_{(0,0),1}(x)$, $\Delta = \partial_1^2 + \partial_2^2$, at the points x with integer coordinates.
Answer
smallest value: [ ]

Problem 4.1.6 Program: Evaluation of a Bivariate Spline
Write a program `p = spline_bivariate(c,n,H)` which evaluates a bivariate spline $\sum_k c_k b_k$ of degree (n, n) and grid width 1 at the points ℓ/H, $\ell \in \mathbb{Z}^2$, within the standard parameter domain.
Answer
sum of values for $n = 2$, $H = 3$, and $c_k = \cos(k_1 k_2)$, $k_\nu = 1 : 10$: [ ]

4.2 Splines on Bounded Domains

Problem 4.2.1 B-Spline Coefficients of a Biquadratic Polynomial
Represent the polynomial $x \mapsto (x_1 - 1)x_2^2$ as linear combination of bicubic tensor product B-splines with grid width $h = 1$.
Answer
coefficient of $b_{(2,1),1}^{(3,3)}$:

Problem 4.2.2 Representation of a Polynomial by a Biquadratic Spline
Which polynomial does the spline $\sum_{k_1} \sum_{k_2} k_1(1 - k_2^2) b_{k,1}^{(2,2)}$ represent?
Answer
largest coefficient of the monomial form $\sum_\alpha p_\alpha x^\alpha$:

4.3 Weight Functions

Problem 4.3.1 Singularity of the Distance Function for an Ellipse
For the ellipse

$$E: \; x^2 + 4y^2 = 1,$$

determine the line segment along which the distance function has a ridge.
Answer
line segment $[-a, a] \times \{0\}$ with $a =$

Problem 4.3.2 Equivalence of R-Functions
Show that the R-function $w(u, v) = u + v + \sqrt{u^2 + v^2}$ and $\max(u, v)$ have for all arguments $u, v \in \mathbb{R}$ the same sign. Moreover,

$$|w(u, v)| \asymp |\max(u, v)| \, .$$

Answer
largest value of the quotient $|w(u, v)|/|\max(u, v)|$:

Problem 4.3.3 Program: Rvachev Operations for Weight Functions Represented by `m`-Files
Write a program `rfct(w,operation,w1,w2)`, which implements Rvachev's method for weight functions represented by MATLAB `m`-files `w1, w2`, by generating an `m`-file `w` according to the specified Boolean operation (union, intersection, or complement).

Problem 4.3.4 Smooth Weight Functions at a Reentrant Corner
If a smooth weight function w vanishes on the negative x- and y-axis and is nonnegative if one of its arguments x or y are positive, then w has at least a fourth order zero at the origin.

4.4 WEB-Splines

Problem 4.4.1 Extension Coefficients for Bilinear WEB-Splines
Determine the extension coefficients $e_{i,j}$ of bilinear WEB-splines for indices j which are adjacent to the array $I(j)$.
Answer
sum of different values:

Problem 4.4.2 Extension Coefficients for Biquadratic WEB-Splines
Compute the extension coefficients $e_{i,j}$ for $j = (0, 1)$ and $i \in I = \{3, 4, 5\} \times \{6, 7, 8\}$.
Answer
largest coefficient:

Problem 4.4.3 Program: Extension Coefficients
Write a program `E = extension_coefficients(J,L,n)`, which computes the array $e_{i,j}$, $i_\nu = \ell_\nu, \ldots, \ell_\nu + n$, of extension coefficients for an outer index $(j_1, \ldots, j_d)$.
Answer
$\max e_{i,j}$ for $J = (1,2,3)$, $L = (9,8,7)$, and $n = 3$: ☐

Problem 4.4.4 Extension Coefficients for Bilinear WEB-Splines
Sketch the domain $D: x_1^2 + x_2^2 < 2$, $x_\nu > 0$, and, for bilinear B-splines b_k with integer knots (grid width 1) the relevant part of the grid. Mark the lower left corners of the supports of the inner and outer B-splines with dots and circles, respectively, and compute the extension coefficients $e_{i,j}$, $j \in J(i)$, for the B-spline b_i with the largest portion of support in D.
Answer
$\max_{j \in J(i)} e_{i,j}$: ☐

Problem 4.4.5 Extension Coefficients for Bilinear WEB-Splines
Sketch the domain $D: 0 < x_1 < x_2 < 2$, and, for bilinear B-splines b_k with integer knots (grid width 1) the relevant part of the grid. Mark the lower left corners of the supports of the inner and outer B-splines with dots and circles, respectively, and compute the extension coefficients $e_{i,j}$, $i \in I(j)$, for $j = (1,0)$.
Answer
$\max_{i \in I(j)} e_{i,j}$: ☐

Problem 4.4.6 Integrity of the Extension Coefficients
Show that the extension coefficients of WEB-splines are integers.

4.5 Hierarchical Bases

Problem 4.5.1 Dimension of a Bivariate Hierarchical Spline Space
Determine the dimension of the bivariate hierarchical spline space with

$$D_\nu = \left(0, 2^{-\nu}\right)^2, \quad 0 \le \nu < \ell \quad (D_\ell = \emptyset),$$

$h = 2^{-\alpha}$ $(\alpha > 0)$, and degree $n = (1,1)$.
Answer
dimension for $\alpha = 1$ and $\ell = 5$: ☐

Problem 4.5.2 Approximation of the Square Root with Piecewise Linear Hierarchical B-Splines
Show that the function $\sqrt{x}$ can be approximated in the maximum norm on the interval $[0,1]$ with error $\le$ `tol` with $\le c\,\texttt{tol}^{-1/2}$ hierarchical linear B-splines.

5 Approximation with Weighted Splines

5.1 Dual Functions

Problem 5.1.1 Dual Functions for Hat-Functions on a Uniform Triangulation
Construct dual functions λ_i for hat-functions B_i on a triangulation consisting of equilateral triangles with edge length h. Choose λ_i as a piecewise linear function which is nonzero only on the support of B_i.
Answer
sum of the values of a piecewise linear dual function λ_i at the middle and one outer vertex for $h = 1$: ☐

Problem 5.1.2 Dual Functions for Univariate Quadratic B-Splines
Construct a quadratic dual function λ_* with support $[3/2-\theta/2, 3/2+\theta/2] \subseteq [1,2]$ for quadratic B-splines b_k, i.e.,

$$\langle \lambda_*, b_k \rangle_0 = \delta_{0,k}, \quad k \in \mathbb{Z}\,,$$

and determine the behavior of $\|\lambda_*\|_0$ as $\theta \to 0$.
Answer
$\|\lambda_*\|_0 \asymp \theta^{-\alpha}$ with α: ☐

Problem 5.1.3 Dual Functions for Linear Bernstein Polynomials
Construct dual functions λ_k, $k = 0, 1$, for the linear Bernstein polynomials $b_0(x) = 1 - x$, $b_1(x) = x$. Determine a constant γ so that

$$|c_k|^2 \leq \gamma \int_0^1 |c_0 b_0(x) + c_1 b_1(x)|^2 \, dx, \quad k = 0, 1\,,$$

for all coefficents c_0, c_1.
Answer
$\max(r, s)$: ☐

Problem 5.1.4 Dual Functions for Monomials
Construct dual functions λ_k for the monomials $b_k(x) = x^k$, $k = 0, 1, 3$, on the interval $[-1, 1]$.
Answer
absolute value of the largest coefficient of the polynomials λ_k: ☐

Problem 5.1.5 Dual Functions for Bilinear B-Splines
Construct dual functions for bilinear B-splines with grid width h, making an ansatz with piecewise bilinear functions with the same support as the corresponding B-splines.
Answer
largest value of the dual functions at the grid points for $h = 1$: ☐

5.2 Stability

Problem 5.2.1 Lower Bound for the Condition Number of the Ritz-Galerkin Matrix for Unstable Bases
Show that

$$\sup_h h^2 \operatorname{cond} G_h = \infty$$

for the Ritz-Galerkin matrix G_h of the Poisson problem on the unit ball and weighted B-splines wb_k, $w(x) = 1 - \|x\|^2$.

Problem 5.2.2 Stability of WEB-Splines in the Maximum Norm
Prove the stability of WEB-splines in the maximum norm:

$$\sup_{x\in D}\left|\sum_{i\in I} c_i B_i(x)\right| \asymp \max_{i\in I}|c_i|$$

for a standard weight function.

5.3 Polynomial Approximation

Problem 5.3.1 Dependence of the Constant in Bramble-Hilbert's Estimate for a Hyperrectangle
Derive the following special case of Bramble-Hilbert's estimate:

$$\inf_c |f-c|_{0,R} \le \text{const(m)}\ \text{diam}(R)\,|f|_{1,R}\,,$$

where R is a hyperrectangle in $\mathbb{R}^m$. In other words, show that the constant is independent of the ratios of the side lengths for hyperrectangles, while in general the estimate might depend on the shape of the domain.

Problem 5.3.2 Dependence of the Constant in Bramble-Hilbert's Estimate on the Domain
Show that the dependence of the constant in Bramble-Hilbert's estimate on the domain is essential by giving a counterexample to the inequality

$$\inf_c |f-c|_{0,D} \le \text{const}\ \text{diam}(D)\,|f|_{1,D}\,,$$

where $D \subset \mathbb{R}^2$ and const neither depends on D nor on f.

5.4 Quasi-Interpolation

Problem 5.4.1 Error of a Bilinear Interpolant for Smooth Functions
By a direct argument, show that the error of a bilinear spline interpolant with grid-width h is of order $O(h^2)$ for smooth functions.

Problem 5.4.2 Schoenberg's Scheme for WEB-Splines
For a standard weight function w of a domain D and a smooth function f which vanishes on ∂D show that

$$f(x) - \sum_{i\in I} f(x_i)B_i(x) = O(h)\,,$$

where x_i are the centers of the interior grid cells, used in the normalization of the WEB-splines.

5.5 Boundary Regularity

Problem 5.5.1 Differentiability of Univariate Quotients
Assume that w is a univariate weight function with $w(0) = 0$, $w'(0) > 0$, and $w(x) > 0$ for $x > 0$. Show that the quotient u/w is ℓ times continuously differentiable on $[0, 1]$ if $u(0) = 0$ and u and w have continuous derivatives up to order $\ell + 1$. Give an example which confirms that the division causes a loss of approximately one order of differentiability.

Problem 5.5.2 Bounded Gradient of a Quotient with R-Functions
Prove that, for a smooth function u which vanishes on the positive coordinate axes, the quotient u/w, $w(x,y) = x + y - \sqrt{x^2+y^2}$, has a bounded gradient on $D = (0,1)^2$.

Problem 5.5.3 Bounds for the derivatives of R-Functions
Show that

$$|\partial^\alpha r| \preceq r^{1-|\alpha|}\,,$$

where $r = \sqrt{x_1^2 + \cdots + x_m^2}$.

5.6 Error Estimates for Standard Weight Functions

No problems for this section.

6 Boundary Value Problems

6.1 Essential Boundary Conditions

Problem 6.1.1 Function on an Annulus with Prescribed Boundary Conidtions
Construct a function u, defined on an annulus

$$D: \ 1 < x^2 + y^2 < 4\,,$$

which satisfies $\partial^\perp u(x,y) = x$ on the inner and $u(x,y) = y$ on the outer boundary.

Problem 6.1.2 Progam: Elimination of Boundary Values and Solution of Poisson's Problem
Write a program `u = poisson_inhomogeneous(f,g,H,n)` which solves the inhomogeneous Poisson problem

$$-\Delta u = f \text{ in } D = (0,1)^2, \quad u = g \text{ on } \partial D\,,$$

for polynomial data ($p(x,y) = \sum_{j,k} p_{j,k} x^{j-1} y^{k-1}$, $p = f, g$) with weighted splines of degree n and grid-width $1/H$ using the FEMB package.
Answer
$\max u$ for $f = g = [1,1,1;\ 1,1,1;\ 1,1,1]$, $H = 10$ and $n = 3$: ☐

Problem 6.1.3 Program: Error of Weighted Spline Approximations for Poisson's Problem
Write a program `[u,error] = bvp_convergence(H,n)` which computes a weighted spline approximation of degree n with grid-width $1/H$ for the Poisson problem

$$-\Delta u = \exp(x^3 y^2 z), \quad (x,y,z) \in D: \ x^2 + y^4 + z^6 < 1\,,$$

with homogeneous Dirichlet boundary conditions. The program evaluates the solution at the centers of the grid cells with width $1/(2H)$ and obtains an error estimate in the maximum norm by comparing with a solution for the fine grid at these points.
Answer
error for $H = 8$ and $n = 2$: ☐

6.2 Natural Boundary Conditions

Problem 6.2.1 Convergence of Richardson's Iteration for the Neumann Problem for Poisson's Equation
Show that the Richardson-Iteration

$$U \leftarrow U + (F - GU)/\|G\|$$

for the Ritz-Galerkin approximation with WEB-splines of the Neumann problem for Poisson's equation,

$$-\Delta u = f \text{ in } D, \quad \partial^\perp u = 0 \text{ on } \partial D\,,$$

($\int_D f = 0$) converges.

Problem 6.2.2 Ritz-Galerkin System Describing Flow in a Channel
For biquadratic B-splines b_k with grid width h, determine the entries

$$g_k = \int_{\Gamma_0} b_k - \int_{\Gamma_1} b_k, \quad \Gamma_\nu = \{\nu L\} \times (0,1),\ k \sim \{0, L\} \times (0,1)\,,$$

of the right side of the Ritz-Galerkin system describing flow in a channel of integer length L.
Answer
$\sum_k |g_k|$ for $h = 1/10$: ☐

6.3 Mixed Problems with Variable Coefficients

Problem 6.3.1 Program: Bivariate Second Order Boundary Value Problem with Variable Coefficients
Write a program `[u,r] = bvp_bivariate(H,n)` which solves the boundary value problem

$$-\operatorname{div}(\mathrm{e}^{xy} \operatorname{grad} u) = 1, \quad u(x,0) = 0\,,$$

on the halfdisc in the first quadrant with radius $1/2$ and center $(1/2, 0)$ using weighted splines of degree n and grid-width $1/H$. The numerical solution u is evaluated at the centers of the grid cells. Moreover, the maximum absolute value r of the residual is computed.
Answer
r for $H = 10$ and $n = 3$: ☐

Problem 6.3.2 Unique Solvability of a Univariate Boundary Value Problem
For which values of the parameters $\alpha, \beta \in \mathbb{R}$ does the boundary value problem

$$u'' = \alpha u, \quad u'(0) = 0,\ u'(1) + \beta u(1) = p\,,$$

have a unique solution for any $p \in \mathbb{R}$?
Answer
for $\alpha = -\pi^2/16$, $\beta \neq$ ☐

Problem 6.3.3 Ellipticity of a Bilinear Form with Variable Coefficients
Show that the bilinear form

$$a(u,v) = \int_D \operatorname{grad} u \ \operatorname{grad} v + \int_D (a \operatorname{grad} u) v$$

is elliptic on $H_0^1(D)$ if $\alpha = \sup_{x \in D} \|(a_1(x), a_2(x), \ldots)\|$ is sufficiently small.

Problem 6.3.4 Hypothesis of the Lax-Milgram Theorem
Determine the variational equations $a(u,v) = \lambda(v)$ for the partial differential equation

$$-\operatorname{div}(\cos(xy) \operatorname{grad} u) + \mathrm{e}^y u = \ln|x|, \quad (x,y) \in D = (-1,1)^2$$

with Dirichlet boundary conditions ($u = 0$ on ∂D). Verify the hypothesis of the Lax-Milgram theorem, i.e., find ellipticity constants c_b and c_e for the bilinear form a on $H_0^1(D)$ and show the boundedness of λ on $L_2(D)$.
Answer
$\|\lambda\|_0$: ☐

Problem 6.3.5 Hypothesis of the Lax-Milgram Theorem
Determine the variational equations $a(u,v) = \lambda(v)$ for the partial differential equation

$$-\mathrm{e}^y u_{xx} - u_{yy} + \cos x\, u = \ln(x^2+y^2), \quad (x,y) \in D:\ r = \sqrt{x^2+y^2} < 1,$$

with Dirichlet boundary conditions ($u(x,y) = 0$ for $r = 1$). Verify the hypothesis of the Lax-Milgram theorem, i.e., find ellipticity constants c_b and c_e for the bilinear form a on $H_0^1(D)$ and show the boundedness of λ on $L_2(D)$.
Answer
$\|\lambda\|_0$: ☐

Problem 6.3.6 Robin Boundary Condition for a Ritz-Galerkin Approximation with Hat-Functions on an Annulus
Consider the Ritz-Galerkin approximation with hat-functions of the boundary value problem

$$-\Delta u = f, \quad \partial^\perp u_{|\Gamma_1} = 0,\ u_{|\Gamma_2} = 0\,,$$

on the annulus bounded by the circles $\Gamma_r : \|x\| = r$ with radii $r = 1, 2$. Describe the modification of the Ritz-Galerkin system, if the natural boundary condition is replaced by

$$\partial^\perp u + 2u = 3 \text{ on } \Gamma_1$$

and all boundary triangles of the triangulation have edge length h.
Answer
largest modification of a matrix or vector entry: ☐ h

6.4 Biharmonic Equation

Problem 6.4.1 Radially Symmetric Solution of the Biharmonic Equation
Determine the radially symmetric solution of the boundary value problem

$$\Delta^2 u = 1 \text{ in } D, \quad u = \partial^\perp u = 0 \text{ on } \partial D\,,$$

with $D:\ r^2 = x_1^2 + x_2^2 < 1$ the unit disc.
Answer
$u(0,0)$: ☐

Problem 6.4.2 Program: Ritz-Galerkin Approximation of a Univariate Fourth Order Problem with Quadratic B-Splines
Write a program `[u,e] = fourth_order(H)` which computes the solution of the boundary value problem

$$u^{(4)} = 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0\,,$$

at the grid points $0, 1/H, \ldots, 1$ as well as the error $e = \max_{0<\ell<1/H} |u_\ell - u_{\text{exact}}(\ell/H)|$.
Answer
error for $H = 32$: ☐

Problem 6.4.3 Ritz-Galerkin Matrix for the Biharmonic Equation and Biquadratic B-Splines
Determine the entries of the Ritz-Galerkin matrix for the biharmonic equation and biquadratic B-splines:

$$g_{j,k} = \int \Delta b_{j,h}^{(2,2)} \Delta b_{k,h}^{(2,2)} \, .$$

Answer
largest entry for $h = 1$: ☐

Problem 6.4.4 Weak Form and Existence for a Biharmonic Equation with Non-standard Boundary Conditions
Determine the weak form of the boundary value problem

$$\Delta^2 u = 0 \text{ in } D, \quad u = 0,\, (\partial^\perp)^2 u = f \text{ on } \partial D\, ,$$

and show the existence of a unique solution for $f \in H^1(D)$ and a domain D with smooth boundary.

6.5 Linear Elasticity

Problem 6.5.1 Strain Tensor for a Radially Symmetric Displacement
Compute the strain tensor $\varepsilon(u)$ for a radially symmetric displacement $u(x) = \varphi(r)x$, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.
Answer
$\sum_{k,\ell} \varepsilon_{k,\ell}$ for $\varphi(r) = r^2$ and $x = (1, 1, 1)$: ☐

Problem 6.5.2 Program: Elasticity Bilinear Form for Hat-Functions on a Tetrahedron
Write a program `G = P_sigma_epsilon_hat(P,lambda,mu)` which computes the $4 \times 4 \cdot (3 \times 3)$ block matrix

$$\int_{[p_1,p_2,p_3,p_4]} \sigma(B_k e_\alpha) : \varepsilon(B_\ell e_\beta)$$

for the hat-functions B_j which correspond to the vertices p_j of a tetrahedron and the unit vectors e_1, e_2, e_3.
Answer
$\sum_{k,\ell,\alpha,\beta} |g_{k,\ell,\alpha,\beta}|$ for the standard simplex with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ and $\lambda = \mu = 1$: ☐

Problem 6.5.3 Relation Between Stress and Strain Tensor
Express the strain tensor ε in terms of the stress tensor σ.
Answer
$\operatorname{trace} \sigma / \operatorname{trace} \varepsilon$ for $\lambda = \mu = 1$: ☐

6.6 Plane Strain and Plane Stress

Problem 6.6.1 Program: Deformation of a Tunnel under a Constant Volume Force
Write a program `d = tunnel(H,n)` which computes the maximal displacement of a concrete tunnel (Young modulus $E = 1$, Poisson ratio $\mu = 1/4$), with cross section bounded by two parabolas and the x-axis, under a constant volume force $f = (0, -1)$. Use splines of degree n and grid-width $1/H$. The parabolas pass through the points $(1/2, 3/4), (1/2 \pm 1/4, 0)$ and $(1/2, 1), (1/2 \pm 1/2)$, respectively, and the structure is fixed at the horizontal boundary ($y = 0$).
Answer
$d_{\max}$: ☐

Problem 6.6.2 Coercivity of the Plane Strain Bilinear Form
Show that the bilinear form

$$\int_D \underline{\varepsilon}(u)\, Q_{\text{strain}}\, \underline{\varepsilon}(v)$$

is coercive on $H_0^1(D) \times H_0^1(D)$.

Problem 6.6.3 Plane Strain Bilinear Form for Pairs of Basis Functions
For given basis functions B_j, B_k and the unit vectors e_1, e_2, determine the entries

$$g_{\alpha,\beta} = \int_D \underline{\varepsilon}'(B_j e_\alpha)\, Q_{\text{strain}}\, \underline{\varepsilon}(B_k e_\beta), \quad 1 \le \alpha, \beta \le 2\,,$$

of the block matrices appearing in the Ritz-Galerkin system for the plane strain model in terms of the integrals $a_{\ell,m} = \int_D \partial_\ell B_j \partial_m B_k$.
Answer
number of integrals making up a matrix entry: ☐

Problem 6.6.4 Program: Deformation of an Eccentric Rotating Disc
Write a program `u = rotating_disc(d,r,H,n)` which computes the displacement of a steel disc (Young modulus $E = 1$, Poisson ratio $\mu = 1/5$) with radius $1/2$ and inner axis with radius r, dislocated by a distance d, under a centrifugal force (proportional to the distance to the center of the axis). Use splines of degree n and grid-width $1/H$.
Answer
maximal absolute value of the components of the dispacement u at the centers of the grid cells for $d = 1/20$, $r = 1/10$, $H = 10$, $n = 3$: ☐

7 Multigrid Methods

7.1 Multigrid Idea

Problem 7.1.1 Program: V-Cycle for a Univariate Model Problem with Hat-Functions
Write a program `W = V_cycle(U,F,alpha)` which implements one step of the v-cycle with hat-functions for the model problem

$$-u'' = f, \quad u(0) = 0 = u(1)\,.$$

Use α Richardson iterations and a coarsest grid with only 1 unknown, i.e., $h_{\max} = 1/2$.
Answer
$(\sum_k w_k^2)^{1/2}$ for $U = (0, \ldots, 0)^{\mathrm{t}}$, $F = (\sin(\pi/32), \ldots, \sin(31\pi/32))^{\mathrm{t}}$, and $\alpha = 4$: ☐

Problem 7.1.2 Grid Transfer for the Fourier Basis with the Univariate Multigrid Restriction Operator
Show that the univariate multigrid restriction operator

$$P^t = \begin{pmatrix} 1/2 & 1 & 1/2 & & & & \\ & & 1/2 & 1 & 1/2 & & \\ & & & & \ldots & \ldots & \ldots \end{pmatrix}$$

leaves the Fourier components corresponding to slow frequencies
$(\theta = \pi, 2\pi, \ldots, (1/(2h) - 1)\pi)$ invariant up to scaling:

$$P^t E_\theta^h = s_\theta E_\theta^{2h}, \quad (E_\theta^h)_k = \sin(\theta k h)\,.$$

Answer
s_θ for $h = 1/64$, $\theta = 16\pi$: ☐

Problem 7.1.3 Representation of Moments for Univariate Hat-Functions
Construct a function r such that

$$r_i = \int_0^1 r b_i\,,$$

for given moments r_i and b_i the univariate hat-functions with support $ih + [-h, h]$, $i = 1, \ldots, 1/h - 1$.
Answer
$\max_x r(x)$ for the construction outlined in the hint, $h = 1$, and $r_i = \delta_{i,k}$: ☐

Problem 7.1.4 Program: Norm of the Univariate Two-Grid Iteration Matrix
Write a program `r = norm_two_grid(h,alpha)` which computes the norm of the two-grid iteration matrix with α Richardson steps for the Ritz-Galerkin approximation of the model problem

$$-u'' = f, \quad u(0) = 0 = u(1)\,,$$

with hat-functions.
Answer
norm for grid width $h = 1/32$ and $\alpha = 4$ Richardson iterations: ☐

7.2 Grid Transfer

Problem 7.2.1 Program: Extension of Bilinear Splines to a Finer Grid
Write a program `V = extend_bilinear(U)` which extends a coefficient vector for bilinear splines to a finer grid.
Answer
$\sum v_k$ for $u_k = k_1 k_2$, $k_\nu = 0, \ldots, 10$: ☐

Problem 7.2.2 Algorithmic Description of the Grid Transfer for Univariate B-Splines
The transfer of univariate B-spline coefficients U to a finer grid (multigrid extension) amounts to duplicating each entry u_k and then forming n-times a simultaneous average of all adjacent entries. Give an analogous description for the multigrid restriction operation.
Answer
largest entry of the restriction of $U = (1, 2, 3, 4)$ for $n = 2$: ☐

Problem 7.2.3 Program: Multigrid Extension for Univariate B-Splines
Write a program `V = extend(U,n)`, which transfers the coefficients of B-splines of degree n on the standard interval $[0, 1]$ to a finer grid.
Answer
$\sum_k v_k$ for $U = (1, 4, 9, \ldots, 81)^t$ and $n = 2$: ☐

Problem 7.2.4 Program: Multigrid Restriction for Univariate B-Splines
Write a program `U = restrict(V,n)`, which transfers the coefficients of B-splines of degree n on the standard interval $[0, 1]$ to a coarser grid.
Answer
$\sum_k u_k$ for $V = (1/1, 1/2, \ldots, 1/9)^\mathrm{t}$ and $n = 3$: ☐

7.3 Basic Algorithm

Problem 7.3.1 Operation Count for a Recursive Algorithm
Consider the algorithm

$U \to V = M(U, k)$
  $V = S(U, k)$
  if $k > 0$
    $V = M(V, k - 1)$
  end

where the function S requires $\le \alpha 2^{\beta k} + \gamma$ operations ($\alpha, \beta, \gamma > 0$). Show that the total number of operations required by M is $\le c 2^{\beta k}$ where the constant does not depend on k.

Problem 7.3.2 Program: Jacobi Iteration for a Ritz-Galerkin System in Sparse Format
Write a program `V = jacobi(U,G,F)` which implements a step $U \to V$ of the Jacobi iteration for the Ritz-Galerkin system $GU = F$. Assume that the matrix G is stored as a $(H + n) \times (H + n) \times (2n + 1) \times (2n + 1)$ array where n denotes the B-spline degree and H the number of grid cells per coordinate direction.
Answer
$\min_{j,k} v_{j,k}$ for $H = 10$, $n = 3$, $u_{j,k} = jk$, and the Ritz Galerkin matrix containing 100 on the diagonal and 1 on the offdiagonals: ☐

7.4 Smoothing and Coarse Grid Approximation

Problem 7.4.1 Program: Smoothing of Richardson's Iteration for a Univariate Model Problem

Write a program `r = richarson_smoothing(F,steps)` which plots the residuals generated by `steps` Richardson iterations for the Ritz–Galerkin discretization with hat-functions of the model problem

$$-u'' = f, \quad u(0) = 0 = u(1)\,,$$

and computes the average reduction factor r of the residuals in the 2-norm.

Answer

r for 10 steps and $F = (\sin(1), \ldots, \sin(31))^{\mathrm{t}}$: ☐

7.5 Convergence

Problem 7.5.1 Convergence of a Power Recursion

Determine the restriction on the parameters $r, s > 0$ for which the sequence $0 = \varrho_0, \varrho_1, \ldots$, generated by the recursion

$$\varrho_{\ell+1} = r + s\varrho_\ell^m \quad (m > 1)$$

converges.

Answer

maximal admissible r if $m = s = 3$: ☐

8 Implementation

8.1 Boundary Representation

Problem 8.1.1 Parametrization of a Segment of a Hyperbola
Determine a rational quadratic parametrization of the segment of a hyperbola, defined implicitly by $xy = 1$, $0 < x, y \leq a$.
Answer
middle weight of a parametrization with $w_0 = w_2 = 1$ for $a = 2$: ☐

Problem 8.1.2 Implicit Representation of a Quadratic Bézier Curve
Derive an implicit representation of the rational Bézier curve with control points and weights

$$\left(\begin{array}{c|c} c_0 & 1 \\ c_1 & w \\ c_2 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1/2 \\ 0 & 1 & 1 \end{array}\right),$$

thereby showing that the curve parametrizes an ellipse.
Answer
ratio of the lengths of the larger and smaller axis: ☐

Problem 8.1.3 Approximation of a Sphere by Bi-Quadratic Bézier Patches
Determine a fully symmetric approximation of the sphere $S : x_1^2 + x_2^2 + x_3^2 = 3$ by 6 bi-quadratic polynomial Bézier patches. The patches should touch the sphere at $(\pm 1, \pm 1, \pm 1)$ (endpoint interpolation) as well as at $(\pm\sqrt{3}, 0, 0)$, $(0, \pm\sqrt{3}, 0)$, $(0, 0, \pm\sqrt{3})$.
Answer
maximum of the third components of the control points: ☐

8.2 Classification of Grid Cells

Problem 8.2.1 Maximizing the Distance of Critical Points to Grid Cell Boundaries
Determine a shift $s \in [0, 1)^m$ such that the distance of given critical points $x^1, \ldots, x^L$ to the shifted grid with cells $(s + k + [0, 1)^m)h$, $k \in \mathbb{Z}^m$, is maximal.

Problem 8.2.2 Sufficient Condition for Inner Grid Cells in Terms of the Weight Function
Let $Q = x_Q + [-1/2, 1/2]^m h$ be a grid cell and $D : w(x) > 0$ be a bounded implicitly defined domain in $\mathbb{R}^m$. If $\sup_{x \in D} |\operatorname{grad} w(x)| \leq c$, derive a lower bound on $\delta = w(x_Q)$ that guarantees that $Q \subseteq \overline{D}$.
Answer
lower bound for δ for $c = h = 1$ and $m = 4$: ☐

Problem 8.2.3 Stability of for Modified B-Spline Classification
Let $D : w > 0$ be a bounded domain with smooth boundary described by a standard weight function w. Modifying the web-spline classification, we can select inner B-splines b_i by requiring that w is positive at the centers of their support. Prove that, also for this alternative definition, each set $\overline{D} \cap \operatorname{supp} b_i$ contains a ball of radius $\geq c(w)h$, which is an essential property for stability.

8.3 Evaluation of Weight Functions

Problem 8.3.1 Program: Distance to a Planar Rational Bézier Curve

Write a program `[r,d] = distance_bezier(C,w,x)` which computes the distance $d = \|x - r(t)\|$ of a point x to a planar rational Bézier curve, parametrized by

$$t \mapsto (r_1(t), r_2(t)) = \frac{\sum_{\nu=0}^{n} (c_{\nu,1}, c_{\nu,2}) \omega_\nu \, \beta_\nu^n(t)}{\sum_{\nu=0}^{n} \omega_\nu \, \beta_\nu^n(t)}, \quad 0 \le t \le 1 \, .$$

Answer

distance for $C = [3,0;\ 0,0;\ 0,2]$, $w = [1;\ 2;\ 3]$, $x = [1,1]$: ☐

Problem 8.3.2 Program: Distance to the Boundary of an Implicitly Defined Domain

Write a program `[x,d] = distance_boundary(w,p,x0,tol)` which computes a closest point x to p on the boundary of a domain $D: \ w > 0$ as well as the distance d of p to ∂D. Use Newton's method and assume that w returns values, gradient, and Hesse-matrix and that the starting guess `x0` is sufficiently accurate.

Answer

distance for $w(x) = x_1^4 - 4x_1^2 + x_2^2 - 1$, $p = (1,0)$, $x_0 = (0,1)$, and $\mathrm{tol} = \mathtt{1e-10}$: ☐

8.4 Numerical Integration

Problem 8.4.1 Numerical Integration of Bilinear Splines Over a Boundary Cell

For $\Omega: \ 0 \le x_1, x_2 \le 1$, $x_2 \le x_1^2$ and bilinear B-splines b_k with grid width 1, determine weights γ_k such that

$$\int_\Omega p(x)\, dx = \sum_k \gamma_k c_k, \quad p = \sum_k c_k b_k \, .$$

Answer

$\sum_k 1/\gamma_k =$ ☐

Problem 8.4.2 Gauß Parameters for a Bivariate Boundary Cell

Using the univariate formula

$$\int_0^1 f \approx \frac{1}{2} f(1/2 - \sqrt{3}/6) + \frac{1}{2} f(1/2 + \sqrt{3}/6) \, ,$$

determine weights γ_ℓ and nodes (x_ℓ, y_ℓ) for integration over the boundary cell

$$\Omega : 1 - 2xy > 0 \, , \ 0 < x, y < 1 \, .$$

Answer

approximation of $\int_\Omega x^y \, dxdy$: ☐

Problem 8.4.3 Dependence of the Accuracy of Gauß Formulas on the Order of Integration
Compute the area of

$$D: y^3 \le x \le h,\ 0 \le y \le h\,,$$

with a piecewise Gauß formula of order 4 using both possible orders of integration and compare the results.
Answer
order of accuracy for the less accurate integration order: ☐

8.5 Matrix Assembly

Problem 8.5.1 Operations for Computing Ritz-Galerkin Integrals
How many operations are needed to generate the Ritz-Galerkin integrals

$$\int_Q \operatorname{grad}(wb_k)\ \operatorname{grad}(wb_\ell)$$

for a grid cell $Q \subset \mathbb{R}^m$ and all relevant pairs of m-variate B-splines of degree n if a numerical integration formula

$$\int_Q f \approx \sum_{\alpha=1}^{p} c_\alpha f(x_1^\alpha, \ldots, x_m^\alpha)$$

with points x^α in the interior of Q is used and the weight function w is a linear combination of B-splines of degree n? Assume that the Taylor form of the univariate B-spline segments has been generated in a preprocessing step so that Horner's scheme is applicable.
Answer
operations for $m = 2$: $c\,pn^4 + O(n^3)$ with $c =$ ☐

Problem 8.5.2 Program: Conversion of a Ritz-Galerkin Array to Sparse Matrix Format
Write a program `A = convert_matrix(G)` which stores a bivariate Ritz-Galerkin matrix, represented in B-spline format by a $(H+n)\times(H+n)\times(2n+1)\times(2n+1)$ array, as a standard sparse matrix labeling the grid positions (k_1, k_2) by a single index $k = k_1 + (k_2 - 1)(H+n)$.
Answer
$\sum_{i,j} i + j + a_{i,j}$ for $H = 4$, $n = 1$ and $g_{i,j,k,\ell} = i + j + k + \ell$: ☐

Part II

Hints

1 Introduction

Problem 1.0.1 Use the program `example_BASIC` of the FEMB-package as a template. Recall the formula $\Delta = \frac{1}{r}(ru_r)_r$ for the Laplace operator in polar coordinates.

2 Basic Finite Element Concepts

2.1 Model Problem

Problem 2.1.1 Note that the test functions are not restricted at the interval endpoints $x = 0$ and $x = 1$. Moreover, use that $\varphi(1)v(1) + \int_0^1 \psi v = 0$ for all v implies $\psi = 0$ as well as $\varphi(1) = 0$.

Problem 2.1.2 For the quadratic energy functional $\mathcal{Q}$ derive a differential equation for the solution u from the inequality

$$\mathcal{Q}(u + tv) \geq \mathcal{Q}(u)\,,$$

which holds for all smooth v with $v(0) = 0$.

Problem 2.1.3 Multiply by a test function v with $v(0) = v(1) = 0$ and integrate over $(0, 1)$ to obtain the weak form of boundary value problem. Guess an appropriate energy functional and check that the characterization of a minimum does indeed lead to the weak form.

Problem 2.1.4 Use the orthogonality of the basis functions and their derivatives:

$$\int_0^\pi \sin(jx)\sin(kx)\,dx = \int_0^\pi \cos(jx)\cos(kx)\,dx = \frac{\pi}{2}\,\delta_{j,k}$$

for $0 < j, k$.

Problem 2.1.5 Note that a minimizer $u \in H$ of the energy functional

$$\mathcal{Q}(u) = \frac{1}{2}a(u,u) - \lambda(u)$$

satisfies the variational equations $a(u, v) = \lambda(v)\ \forall v \in H$.

Problem 2.1.6 Recall that $\int b_j b_k = h s_{j-k}$ where s_{-1}, s_0, s_1 are the values of the cubic B-spline at the interior knots $1, 2, 3$.

Problem 2.1.7 In deriving the variational equations note that $\int_D \ldots = 2\pi \int_0^1 \ldots\, r\,dr$ for the unit disc D. Moreover, observe that the integrals $\int_0^1 \exp(r^2)B_k(r)\,r\,dr$ can be computed recursively.

2.2 Mesh Based Elements

Problem 2.2.1 Since the support $[kh - h, kh + h]$ of the hat-functions B_k, $k = 1, \ldots, 1/h - 1$, consists only of two grid intervals, the Ritz–Galerkin matrix is tridiagonal.

Problem 2.2.2 The hat-functions B_i, B_j, B_k, which are nonzero on the triangle T, contribute the submatrix

$$\int_T \operatorname{grad} B_m \operatorname{grad} B_{m'}, \quad m, m' \in \{i, j, k\}\,.$$

The gradients are constant on T and their values can be determined from the derivatives in the directions of the triangle edges.

Problem 2.2.3 Use that B_m as well as f are linear on the edges of the tetrahedron to compute the values of fB_m at the midpoints $(p_m + p_{m'})/2$.

Problem 2.2.4 Compute the contributions to the L_2-norm separately for each triangle by transformation to the standard triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Problem 2.2.5 Use an auxiliary array E to avoid double vertices. The entry $e_{j,k}$ contains the vertex number of the midpoint of the edge $[(p_{j,1},p_{j,2}),(p_{k,1},p_{k,2})]$.

Problem 2.2.6 Use the geometric definition of the gradient; direction and length can be determined without computations. Note that by symmetry all off-diagonal entries have the same value.

Problem 2.2.7 Count the triangular Lagrange functions which form a basis. Use Euler's formula $t-e+v=1$ to determine the number of edges e.

Problem 2.2.8 Write a recursive program by decomposing the Lagrange function into appropriate linear factors (quotients of determinants). Use the factorization to reduce the degree.

Problem 2.2.9 Consider the images of the line segments $\{x\}\times[0,1]$.

Problem 2.2.10 Use the quadrature formula

$$\int_T p = \frac{\operatorname{area} T}{3}\sum_{k\sim T} p(e_k)$$

with $e_k, k\sim T$, the midpoints of the edges of the triangle T.

2.3 Sobolev Spaces

Problem 2.3.1 Compute first the univariate sine coefficients $a_\ell = 2\int_0^1 \varphi(t)\sin(\pi\ell t)\,dt$ for the functions $\varphi(t)=1$ and $\varphi(t)=t$.

Problem 2.3.2 $\partial_\nu r = x_\nu/r$.

Problem 2.3.3 Show that the homogeneous boundary conditions are not compatible with the differential equation. Moreover, prove that the sine/cosine expansion of the first order partial derivatives is absolutely convergent.

Problem 2.3.4 Compute, e.g., $\partial_1 f(x)$ for $x\neq(0,0)$. To derive the identity $\int \partial_1 f\varphi = -\int f\partial_1\varphi$, split the integrals over the sets B_ε and $D\backslash B_\varepsilon$ where B_ε is a small disc with center $(0,0)$ and D contains the support of the test function φ.

Problem 2.3.5 Consider a sequence of functions $\varphi_\ell\in H_0^1(D)$ which converge to the constant function, thereby concluding that this function, which obviously belongs to $H^1(D)$, is not an element of $H_0^1(D)$. Use the Poincaré-Friedrichs inequality.

2.4 Abstract Variational Problems

Problem 2.4.1 The derivative of the function $v\in H_0^1(-1,1)$ has a jump discontinuity at $x=0$. Take this into account by integrating the scalar product $\langle u,v\rangle$ by parts, separately on the subintervals $(-1,0)$ and $(0,1)$.

Problem 2.4.2 Derive a partial differential equation for the representing function and solve it using polar coordinates.

Problem 2.4.3 For H_0^1, the Sobolev semi-norm is a norm in view of the Poincare-Friedrichs inequality. Write the integrand as $\operatorname{grad} u\, A \operatorname{grad} v$ and determine the eigenvalues of A to obtain optimal bounds.

Problem 2.4.4 Determine first the minimum and maximum of x_1x_2 on D.

Problem 2.4.5 The Ritz-Galerkin projection p satisfies the variational equations

$$a(p, B_k) = \lambda(B_k)$$

with B_k a basis of the approximating finite element space.

Problem 2.4.6 The convergence rate is optimal if the spectral radius of the iteration matrix is smallest.

Problem 2.4.7 Verify the hypotheses of the Lax-Milgram theorem. Use $u(x) = \int_0^x u'(y)\,dy$ for $u \in H_0^1(0, 1)$.

2.5 Approximation Error

Problem 2.5.1 By the mean value theorem, in every interval $D_k = kh + [0, h]$, there exists a point x_k with $u'(x_k) - u_h'(x_k) = 0$. Use this fact to estimate $\int_{D_k} |u' - u_h'|^2$, noting that u_h' is constant on D_k with vanishing second derivative.

3 B-Splines

3.1 The Concept of Splines

Problem 3.1.1 Denote by $[p]_x = p(x^+) - p(x^-)$ the jump of a function p at x and represent the spline space S as the kernel of the linear map

$$L : p \mapsto ([p]_h, [p']_h, [p]_{2h}, [p']_{2h}, \ldots)$$

between appropriate vector spaces P and V. Use that $\dim P = \dim(\ker L) + \dim(\operatorname{im} L)$.

Problem 3.1.2 Consider the Taylor polynomials on adjacent intervals $[k-1,k]h$ and $[k,k+1]h$ and eliminate the unknown derivatives $p''(kh)$, $p'''((k \pm 1)h)$ in terms of the Hermite data p_ℓ, p'_ℓ, $\ell = k-1, k, k+1$.

3.2 Definition and Basic Properties

Problem 3.2.1 Differentiate the recursion for the derivative of b^n.

Problem 3.2.2 Use induction on n and the recursion for the derivative of a B-spline.

3.3 Recurrence Relation

Problem 3.3.1 Use the recurrence relations for b^n and $(b^n)'$.

3.4 Representation of Polynomials

Problem 3.4.1 Use the tabulated expressions of the B-spline segments.

Problem 3.4.2 Express p as linear combination of the monomials $x \mapsto (x-\ell)^n$, $\ell = 0, \ldots, n$, and apply Marsden's identity. Use the Matlab functions `polyval` and `polyfit`.

Problem 3.4.3 Obtain a first formula for c_k with the aid of Marsden's identity. Derive some qualitative properties of the coefficients (degree, symmetry, etc.). Then, use induction on n or, alternatively, a computer algebra system.

Problem 3.4.4 Use Marsden's identity to obtain representations for the monomials x^ℓ.

Problem 3.4.5 Express b as a linear combinations of B-splines, and, by considering intervals adjacent to the support of b, conclude that all B-spline coefficients must be zero.

3.5 Subdivision

Problem 3.5.1 The grid width h has no effect on the algorithm.

Problem 3.5.2 Show first that each subdivision step reduces the maximal difference of adjacent coefficients at least by a factor 1/2.

Problem 3.5.3 By scaling, one may assume that $h = 1$. First, consider the subdivision of piecewise constant splines, i.e., the identity

$$\sum_k c_k \, b^0(x-k) = \sum_\ell c_\ell^0 \, b^0(mx-\ell)\,.$$

Then, raise the degree by forming the integral average

$$f \mapsto \int_0^1 f(\cdot - t)\, dt$$

on both sides, leading to an algorithm for successively computing the refined coefficients $c_\ell^1, c_\ell^2, \ldots$ for cardinal splines of degree 1, 2, ...

3.6 Scalar Products

Problem 3.6.1 Use the formulas for s_k^n and d_k^n and that the B-splines form a partition of unity.

Problem 3.6.2 Repeatedly apply two steps of the recurrence relation for B-splines, exploiting also their symmetry with respect to the midpoint of their support.

4 Finite Element Bases

4.1 Multivariate B-Splines

Problem 4.1.1 Use symmetry, the formula for univariate B-splines on the first knot interval,

$$b^n(x) = x^n/n!\,,$$

and that the B-spline values sum to 1.

Problem 4.1.2 First, determine the values and derivatives of the univariate B-spline $b^3_{0,1}$.

Problem 4.1.3 By definition, $b^{(2,2)}_{(0,0),1}(x) = b^2(x_1)b^2(x_2)$, where b^2 is the univariate B-spline with knots $0, 1, 2, 3$.

Problem 4.1.4 Use the tabulated values of scalar products of univariate B-splines and their derivatives. Moreover, note that the row values sum to 0.

Problem 4.1.5 First, compute the values and second derivatives of the univariate cubic B-spline at the knots $1, 2, 3$ and recall that $b^3_{(0,0),1}$ is a product of univariate B-splines.

Problem 4.1.6 First, evaluate the univariate B-spline b at the points $0, 1/H, \ldots, n+1$. Then, accumulate the array p by noting that $c_k b(j_1/H)b(j_2/H)$ contributes to the value of the spline

$$\sum c_k b^n(x_1 - k_1)(b^n(x_2 - k_2)$$

at $x = \ell/H$ with $\ell = j + kH$. In loops over j_ν, the corresponding array positions p_ℓ can be updated simultaneously. Finally, clip the values outside the standard parameter domain.

4.2 Splines on Bounded Domains

Problem 4.2.1 Use Marsden's identity and the product form of the polynomial.

Problem 4.2.2 Use the univariate Marsden identity to find representations of the monomials $1, t, t^2$ and form appropriate linear combinations for the x_1- and x_2-components.

4.3 Weight Functions

Problem 4.3.1 By symmetry the line segment has the form $[-a, a] \times \{0\}$. It consists of intersections of normals at points on opposite sides of the ellipse.

Problem 4.3.2 Use polar coordinates $(u, v) = r(\cos t, \sin t)$ and l'Hospital's rule to estimate the quotient $|w(u, v)|/|\max(u, v)|$.

Problem 4.3.3 Use the Matlab-commands `switch`, `strcat`, `fprintf`, `fopen`, and `fclose`.

Problem 4.3.4 Use polar coordinates $(x, y) = r(\cos t, \sin t)$ and consider successively the Taylor polynomials of w of order 1, 2, and 3.

4.4 WEB-Splines

Problem 4.4.1 By translation invariance and symmetry, one may assume $I(j) = \{0, 1\}^2$ and has to consider only $j = (2, 1)$ and $j = (2, 2)$.

Problem 4.4.2 Observe that $e_{i,j} = e_{i_1,j_1} e_{i_2,j_2}$ and that the array $(e_{i,j})_{i\in I}$ only depends on $j - \ell$ where ℓ is the lower left corner of I.

Problem 4.4.3 Use that $e_{i,j} = \prod_{\nu=1}^{d} e_{i_\nu,j_\nu}$ and that the univariate extension coefficients are values of Lagrange polynomials which can be computed with the aid of the Matlab functions `polyfit` and `polyval`.

Problem 4.4.4 The extension coefficients are values of the bilinear Lagrange polynomial which equals 1 at the grid point i.

Problem 4.4.5 The extension coefficients $e_{i,j}$ are the values at j of the bilinear Lagrange polynomials corresponding to the 2×2–array $I(j)$.

Problem 4.4.6 Because of the product form of $e_{i,j}$ it suffices to consider univariate Lagrange polynomials. Express the fractions in terms of binomial coefficients.

4.5 Hierarchical Bases

Problem 4.5.1 The number of relevant B-splines b_{k,h_ν} on the ν-th level is determined by the sets D_ν and $D_{\nu+1}$; $\overline{D} \cap \operatorname{supp} b_{k,h_\nu}$ must be contained in $\overline{D}_\nu$, but not in $\overline{D}_{\nu+1}$.

Problem 4.5.2 Use piecewise linear interpolation with the error bound

$$\min(\sqrt{a+h}, a^{-3/2}h^2/8)$$

on a grid interval $[a, a+h]$. Denote the break points of the hierarchical partition by $a_\ell = k_\ell 2^{-\ell}$ with k_ℓ even, i.e., the partition has the grid width $2^{-\ell}$ between $a_{\ell+1}$ and a_ℓ, and derive bounds for the integers k_ℓ.

5 Approximation with Weighted Splines

5.1 Dual Functions

Problem 5.1.1 Use a symmetric ansatz, i.e., λ_i is determined by the value α at the middle vertex of the support of B_i and the value β at the 6 neighboring vertices. Use the quadrature formula

$$\int_T p = \frac{\operatorname{area} T}{3} \sum_{k \sim T} p(e_k)$$

with e_k the edge midpoints of the triangle T, for integrating quadratic polynomials.

Problem 5.1.2 Use the ansatz $\lambda_*(x) = r - s(x - 3/2)^2$. Then, by symmetry, only the scalar products with two B-splines need to be considered. Use computer algebra for the integrations and solution of the resulting linear system for the parameters r and s.

Problem 5.1.3 Make the ansatz $\lambda_0(x) = r(1-x) + sx$ and note that, by symmetry, $\lambda_1(x) = s(1-x) + rx$. Then, the requirement $\int_0^1 \lambda_j(x) b_k(x)\, dx = \delta_{j,k}$ leads to a linear system for the parameters r, s. Use the orthogonality of the dual functions and the Cauchy-Schwarz inequality to derive the estimate.

Problem 5.1.4 Make the ansatz $\lambda_k(x) = \alpha_k + \gamma_k x^2$ for even k and $\lambda_k(x) = \beta_k x$ for odd k. Then, determine the coefficients from the orthogonality conditions $\int_{-1}^1 \lambda_j(x) b_k(x)\, dx = \delta_{j,k}$.

Problem 5.1.5 By symmetry and translation invariance it suffices to construct a single dual function with support $[-h, h]^2$. Use the values at the grid points $\{-h, 0, h\}^2$ as parameters and note that the integrals need to be computed for one square only.

5.2 Stability

Problem 5.2.1 The condition number is the quotient of the maximum and the minimum of the Raleigh quotient UG_hU/UU, where u_k are the coefficients of the weighted B-splines. Choose grids which have cells with very small intersections with the unit ball. Do not aim for sharp bounds; crude estimates of the Raleigh quotient for appropriate unit vectors U are sufficient.

Problem 5.2.2 By scaling you may assume $h = 1$. To relate the nontrivial upper bound for $|c_i|$ to the standard stability result for B-splines, recall that $|w(x_\ell)/w|$ is bounded on a subcell Q'_ℓ with center x_ℓ and width $1/(2\sqrt{m})$. Moreover, note that the quotients $|w(x_j)/w(x_k)|$ are uniformly bounded as well for B-spline pairs b_j, b_k with some common support.

5.3 Polynomial Approximation

Problem 5.3.1 Transform R to a hypercube with side length $h = \operatorname{diam}(R)$.

Problem 5.3.2 Consider a sequence of U-shaped domains

$$D = (-2, 2) \times (0, 1) \setminus (-1, 1) \times (\varepsilon, 1)$$

with $\varepsilon \to 0$ and functions $f : (x_1, x_2) \mapsto rx_1/(1 + |rx_1|)$ with $r = r(\varepsilon)$.
Note, that the best L_2-approximation to f by constants is 0.

5.4 Quasi-Interpolation

Problem 5.4.1 Consider $[0,h]^2$ as a typical grid cell. Restricted to this cell, the bilinear spline is a bilinear polynomial which interpolates at the corners and is linear on the edges of $[0,h]^2$. Estimate first the partial derivatives of the error on the edges, then the gradient on the entire grid cell, and finally the interpolation error.

Problem 5.4.2 Prove first that $w = \sum_i w(x_i)B_i$ by recalling that linear combinations of WEB-splines can represent weighted polynomials and considering the identity on inner grid cells. Then, write the error in the form $\sum_i (fw(x_i)/w - f(x_i))B_i$.

5.5 Boundary Regularity

Problem 5.5.1 First, consider the quotients $u(x)/x$ and $w(x)/x$.

Problem 5.5.2 Represent u and u_x by integrating u_x and u_{xy}, respectively, and, as a consequence, show that that $u(x,y) = xyv(x,y)$, where v is smooth on $\overline{D}$. To analyze the quotient, use polar coordinates in conjunction with l'Hospital's rule.

Problem 5.5.3 Note that $\partial_\nu r = x_\nu / r$ and show by induction that the higher order partial derivatives are sums of products $r^{\beta_0} x_1^{\beta_1} \cdots x_m^{\beta_m}$ with $\beta_0 + \beta_1 + \cdots + \beta_m = 1 - |\alpha|$.

5.6 Error Estimates for Standard Weight Functions

No problems for this section.

6 Boundary Value Problems

6.1 Essential Boundary Conditions

Problem 6.1.1 Use the ansatz $u = wf + g$, where w is a standard weight function for D with $\partial^\perp w = 1$ on the inner boundary.

Problem 6.1.2 Transform the problem to standard homogeneous form by setting $u = g + v$. Use the weight function $w(x) = x_1(1 - x_1)y_1(1 - y_1)$.

Problem 6.1.3 Transform the domain to fit into the standard bounding box $[0, 1]^3$, and use the program `bvp_3d` of the `FEMB`-package.

6.2 Natural Boundary Conditions

Problem 6.2.1 Note that $\ker G = \operatorname{span} V^0$ with $v_i^0 = 1\ \forall i$. Represent the error as linear combination of the eigenvectors $V^0, V^1, \ldots$ of the iteration matrix $S = \operatorname{id} - G/\|G\|$.

Problem 6.2.2 Only B-splines b_k with $k_1 \in \{-2, -1\}$ or $k_1 \in \{L/h - 2, L/h - 1\}$ do not vanish on both boundaries Γ_ν. You may consider $k_1 = -1$, $k_2 \geq 0$ as a typical case. Essentially you have to compute an integral over a univariate quadratic B-spline which can be expressed in terms of cubic B-splines.

6.3 Mixed Problems with Variable Coefficients

Problem 6.3.1 Use the program `bvp_2d` of the `FEMB`-package.

Problem 6.3.2 Consider three cases, depending on the sign of α.

Problem 6.3.3 Use the Cauchy-Schwarz inequality to estimate $\int (a \operatorname{grad} u)v$ and the inequality $\|u\|_0 \preceq |u|_1$ of Poincaré-Friedrichs.

Problem 6.3.4 $\int_0^1 \ln^2 x\, dx = 2$.

Problem 6.3.5 $\int_0^1 \ln^2 r\, r\, dr = 1/4$.

Problem 6.3.6 Use Simpson's rule

$$\int_0^h p = h(p(0) + 4p(1/2) + p(1))/6, \quad \operatorname{degree} p \leq 2\,,$$

for computing the boundary integrals.

6.4 Biharmonic Equation

Problem 6.4.1 Recall that $\Delta = \frac{1}{r}\partial_r r \partial_r$ and note that the general radial solution of the differential equation cannot contain terms which are singular for $r = 0$.

Problem 6.4.2 The exact solution is a quartic polynomial.

Problem 6.4.3 By scaling you need to consider only the grid width $h = 1$. Note that $b_{k,1}^{(2,2)}(x_1, x_2) = b^2(x_1 - k_1)b^2_{x_2 - k_2}$ and use computer algebra to compute the integrals of the B-spline derivatives.

Problem 6.4.4 The invariance of the Laplace operator under orthogonal coordinate transformations implies that $\Delta u = f$ on ∂D. Moreover, by elliptic regularity, $\|u\|_2 \preceq \|\Delta u\|_0$ since $u = 0$ on ∂D.

6.5 Linear Elasticity

Problem 6.5.1 By the chain rule, $\partial_\nu r = x_\nu / r$.

Problem 6.5.2 The integrand is a constant which can be determined from the gradients of the hat-functions. Compute the gradients by considering appropriate directional derivatives.

Problem 6.5.3 First, derive a relation between the traces of the two tensors.

6.6 Plane Strain and Plane Stress

Problem 6.6.1 Use the plane strain model and the program `elasticity_2d` of the `FEMB`-package.

Problem 6.6.2 In view of the inequality of Poincare-Friedrichs,

$$|u|_1 = \left(\int_D |\partial_1 u_1|^2 + |\partial_2 u_1|^2 + |\partial_1 u_2|^2 + |\partial_2 u_2|^2 \right)^{1/2}$$

is a norm on $H_0^1(D) \times H_0^1(D)$. Moreover, by a density argument, it suffices to consider smooth functions, which is convenient for integrating by parts.

Problem 6.6.3 Use that $\tilde{\varepsilon} Q \varepsilon = \sum_{\ell,m} q_{\ell,m} \tilde{\varepsilon}_\ell \varepsilon_m$.

Problem 6.6.4 Use the plane stress model and the program `elasticity_2d` of the `FEMB`-package.

7 Multigrid Methods

7.1 Multigrid Idea

Problem 7.1.1 Implement multiplications with the Ritz–Galerkin matrix and the grid transfer matrices as simple vector operations.

Problem 7.1.2 Use the trigonometric formula

$$\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta\,\pm\,\cos\alpha\sin\beta\,.$$

Problem 7.1.3 Use the ansatz $r = \sum_k r_k \lambda_k$, where λ_k is piecewise linear with the same support as b_k, and observe that $\int \lambda_k b_i = \delta_{i,k}$.

Problem 7.1.4 Starting from an approximation U, denote the results after α Richardson and one two-grid steps by V and W, respectively. Then,

$$V - U_* = S^\alpha (U - U_*), \quad W - U_* = T(U - U_*)\,,$$

where U_* is the exact solution of the Ritz-Galerkin system and S and T are the Richardson and two-grid iteration matrices.

7.2 Grid Transfer

Problem 7.2.1 Use bilinear interpolation, noting that the coefficients correspond to values at the grid points.

Problem 7.2.2 Describe the duplication and averaging by matrix operations. Then give an algorithmic description for the multiplications with the transposed matrices.

Problem 7.2.3 Using a loop over the binomial coefficients, the entries of U affect every second entry of V. Note, that not all of the refined coefficients are relevant.

Problem 7.2.4 Using a loop over the binomial coefficients, every second entry of V affects the entries of U. For full vectorization, it is convenient to pad V by zeros before applying the restriction operation.

7.3 Basic Algorithm

Problem 7.3.1 Derive a recursion for the number of operations required by M.

Problem 7.3.2 Note that the matrix diagonal corresponds to the entries $G(:,:,n+1,n+1)$. For an efficient implementation, it is convenient to pad the vector $U(1:H+n, 1:H+n)$ by zeros and loop over the third and fourth indices of G.

7.4 Smoothing and Coarse Grid Approximation

Problem 7.4.1 The residual is generated as part of a Richardson step.

7.5 Convergence

Problem 7.5.1 Illustrate the recursion graphically.

8 Implementation

8.1 Boundary Representation

Problem 8.1.1 Choose $w_0 = w_2 = 1$ and make a symmetric ansatz for the control points c_0, c_1, c_2. Use that c_1 is the intersection of the tangents at the endpoints. Determine the middle weight w_1 by evaluating the parametrization at the midpoint.

Problem 8.1.2 Write the parametrization in the form

$$x = \sum_{\nu=0}^{2} \gamma_\nu(t) c_\nu \,,$$

express γ_ν as linear functions of (x_1, x_2), and observe that $\gamma_1^2 = w_1^2 \gamma_1 \gamma_2$.

Problem 8.1.3 By symmetry, only the top patch needs to be determined. Use symmetry and the endpoint interpolation property to determine the patch boundaries. Finally, evaluate the patch at the midpoint to determine the middle control point.

8.2 Classification of Grid Cells

Problem 8.2.1 Consider each component separately and determine s_ν by maximizing

$$\min_{\ell} \min_{k_\nu \in \mathbb{Z}} |x_\nu^\ell - (s_\nu + k_\nu) h \bmod h| \,.$$

Problem 8.2.2 Use the formula $w(y) - w(x) = \int_0^1 \operatorname{grad}(x + t(y-x))(y-x)\, dt$.

Problem 8.2.3 Show first that for any point $x \in \partial D$ and any $r < r_{\max}$ there exists a ball $B_{x,r} \subset D$ with radius r and center with distance $2r$ from x which is contained in D. To this end use bounds on the gradient and Hesse matrix of w and a quadratic Taylor approximation of w near the boundary.

8.3 Evaluation of Weight Functions

Problem 8.3.1 The closest point $r(t)$ to x is either one of the endpoints $r(0)$ or $r(1)$ or satisfies

$$\varphi(t) = \langle x - r(t), r'(t) \rangle = 0 \,.$$

Generate the numerator of φ by interpolating sufficiently many values of the scalar product. Use the Matlab programs `polyfit` and `roots`.

Problem 8.3.2 Note that a closest point is characterized by the equation $p - x = \lambda \operatorname{grad} w(x)$.

8.4 Numerical Integration

Problem 8.4.1 The relevant bilinear B-splines are b_k, $k \in \{-1, 0\}^2$.

Problem 8.4.2 Determine the intersections of $\Gamma : 1 - 2xy = 0$ with the boundary of $(0,1)^2$ and partition Ω accordingly. Map the tensor product Geuß formula to the resulting subdomains.

Problem 8.4.3 The scaled univariate Gauß formula $\int_0^h f \approx h(f(1/2 - \sqrt{3}/6) + f(1/2 + \sqrt{3}/6))/2$ is used in both coordinate directions. Note, that the set D has to be split at corner points.

8.5 Matrix Assembly

Problem 8.5.1 Note that, in view of symmetry, only roughly half of the $(n+1)^m \times (n+1)^m$ relevant B-spline pairs need to be considered. Split the computation into several steps: the evaluation of univariate B-splines and their derivatives, the computation of the m-variate B-splines and their gradients, the generation of w and $\operatorname{grad} w$, and finally the numerical integration.

Problem 8.5.2 Recall that for an array element g_{k_1,k_2,s_1,s_2} the indices s_ν specify the offset from the main diagonal which corresponds to $s = (n+1, n+1)$. Use the Matlab commands `reshape`, `ndgrid`, and `spdiags`.