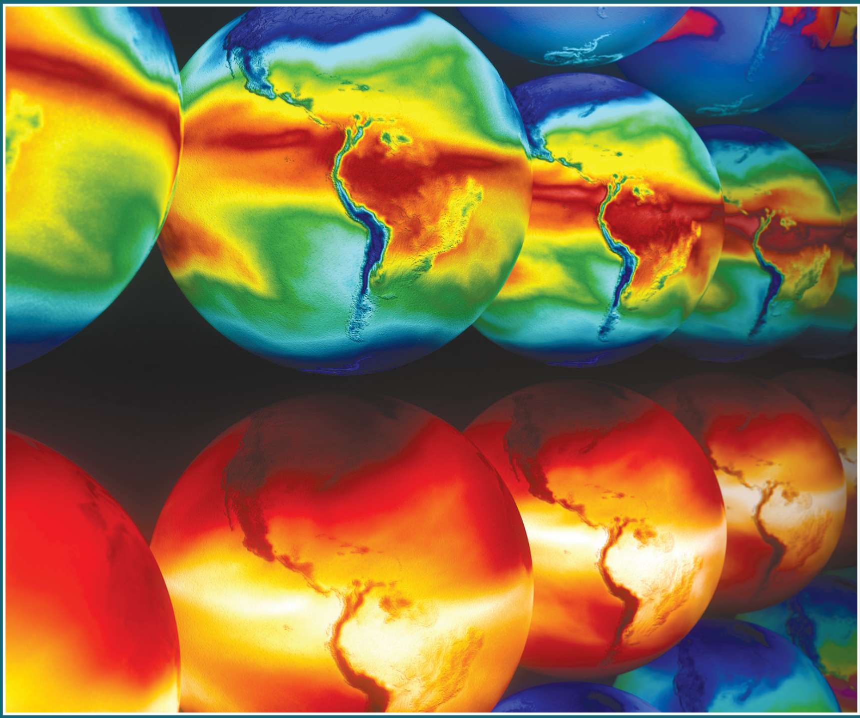
Climate Modeling for Scientists and Engineers

Supplemental Lectures



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Mathematical Modeling and Computation

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Cover art: Visualization of time dependent fields of the community climate system model. Courtesy of Jamison Daniel. This research used resources of the Oak Ridge Leadership Computing Facility at the Oak Ridge National Laboratory, which is supported by the Office of Science of the U.S. Department of Energy under contract no. DE-AC05-00OR22725.

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Supplemental Lectures Contents

Preface

The lectures collected here are supplementary material for the text book *Climate Modeling for Scientists and Engineers*. The purpose of the lectures is to give a more detailed look at a topic that is active in research and inquiry. The idea of each is to lead the student from something they know, or should know, into material that is not well known or not well understood. Most of the lectures follow a common format, presenting a mathematical theorem or construction that we are certain about and then exploring the implications, scientific, social, or otherwise, of these facts.

The students who have taken my courses often express a preference for the wide ranging, and somewhat speculative, lectures to the presentation of governing equations and methods in the main part of the course. Of course, there is not a good way to test on some of this material. It is a source, however, for projects that can launch students into a research mode very quickly. No exercises are specified in the material, keeping with the notion that the student's imagination and curiosity is, hopefully, being stimulated.

The first lecture is on the History of Climate Modeling as told from the perspective of a U.S. Government sponsored (more specifically, Department of Energy sponsored) research community. The Community Climate System Model is the ongoing effort of this community, and as an open community it has had input and cross fertilization from all the major national and international research groups. One might argue that other climate modeling groups deserve more credit and discussion, particularly the group at UCLA and the NOAA Geophysical Fluid Dynamics Laboratory (GFDL). Others are better equipped to discuss and document their contributions, so I can only apologize for my lack of a comprehensive history. On most fronts, the CCSM community was not far behind, if they were not leading, the contributions of these other groups.

Many of the lectures expose some of my personal research interests in methods and mathematics as well as frame a broader discussion of climate change and the role of simulation in the energy debates. Again, the lectures are not meant to be comprehensive in presenting everything that has been or is going on in the field. The student can discover this for themselves very quickly by engaging with the ideas and emerging research literature.

Chapter 1

History of Climate Science Discovery

The story of the discovery of the earth's climate must be understood against the background of geographic exploration and technological progress. Since many of the earliest measurements of the earth were made on voyages of discovery, a rich interplay exists between mapping the globe and understanding the scope of phenomena of weather and climate. The trade winds that carried Columbus across the Atlantic are part of the Hadley (1735) cell, a feature of earth's general circulation. The problem of longitude was a historically significant dilemma for the development of navigation and reliable commerce on the global scale; the wristwatch was a technological spin-off. Before longitude could be measured accurately by keeping track of time and local sunrise, the deviation of earth's magnetic north from true north was used with maps of the earth's magnetic field to determine position. Actually, this is still used today in navigation where GPS has not taken over. Since the earth's magnetic field changes over time, the mapping of geophysical quantities has a long and important role in the history of science and commerce.[1]

1.1 ▪ Classical Notes

Since climate science is the long time, statistical variant of meteorology, it is worth noting that Aristotle wrote a treatise on meteorology. To him this included those extraterrestrial visitors we know as meteors, possibly also UFOs and any other visitors from outer-space. It also included earthquakes as these were thought to be caused by wind. This raises an interesting question of where the earth's atmosphere actually does end and what the bounds for the discipline of climate science are. Clearly, the star we orbit has a lot to do with the weather and temperature of different parts of the earth.[2]

[1]The U.S. Commerce Department hosts the National Oceanographic and Atmospheric Administration (NOAA) which operates the weather service.

[2]In Aristotle's treatise, Meteorology, Book II, Chapter 4, the reason for the winds is traced to the "sun in its circular course draws up by its heat the moist evaporation." It also draws out another dry form of evaporation "sort of like smoke." The moist evaporation causes rain, and the dry evaporation causes the winds. We might translate these notions into latent heat and sensible heat fluxes and acknowledge that differential heating of the earth as a cause of the winds and weather has long been understood. Aristotle argues that the winds cause earthquakes. He mentions climate of habitable and inhabitable regions; these regions have a zonal structure on the earth when viewed from the Mediterranean.

1.2 ▪ Tuning Up with Modern Physics

It didn't go unnoticed that the tides were related to the phases of the moon, though it wasn't until Laplace (1749–1827) extended Newton's gravitational theory with the mathematical theory of partial differential equations (calculus in multiple dimensions) to formulate and solve the Laplace tidal equation,[3] that a full explanation of tides was known. The dynamics of the atmosphere followed a similar unfolding by Newtonian physics and mathematics. Vilhelm Bjerknes (1895),[4] following the link between fluid dynamics and thermodynamics, formulated the basic equations that form the basis of climate models and subsequent discovery of the the fundamental Rossby (1898–1957) waves that carry weather systems. Rossby, a student of Bjerknes, eventually became the director of research at the U.S. Weather Bureau. The modern weather service and numerical weather prediction was developed by Jule Charney, who was a correspondent of Rossby. Charney also founded the National Center for Atmospheric Research (NCAR).

The full theory of the basic equations is even yet beyond our mathematical understanding (see the Clay Mathematics Challenge for the existence and uniqueness of solutions to the Navier–Stokes equations[5]). The development of practical weather forecasting is grounded by the physics and flow equations, technology for long-distance data gathering (e.g., the telegraph), and the development of the computer.

1.3 ▪ Weather Forecasting and Climate

The first numerical weather forecast was produced in 1950 by a team of mathematicians and scientists using the ENIAC computer at Princeton. John Von Neumann, a Manhattan project member and possibly the greatest mathematician of the 20th century, was on this team. He is known for the systemization of quantum mechanics using operator theory as well as the design of modern computer architecture. The first forecast was based on a 24-hour solution of the barotropic vorticity equation initialized from weather station data. (The weather service grew out of this effort.) In 1963 the meteorologist Ed Lorenz developed the theory of deterministic chaos that explains why the weather and climate problems are hard. The theory is illustrated by the flapping of a butterfly wing in China that then affects the weather in North America. Lorenz traced the difficulty to the equations of fluid motion and sparked the area of dynamical systems and chaos theory. Weather, as the particular path of the earth's dynamical system, cannot be precisely predicted for over about 10 days. Lorenz suggested the use of ensembles to define probabilities that can be predicted on longer time scales. Climate prediction is thus not weather prediction but rather the prediction of weather statistics. The global mean surface temperature is the quantity always implicated in discussions of global warming.

Von Neumann realized early on that weather and climate are the relatively easy problems, weather prediction depending on the initial conditions and climate prediction on the asymptotic properties of the system. The midrange time scales of 10 days to 10 years are the hardest to capture and, in fact, may be impossible to predict. So it was somewhat surprising and a great advance when climatologist Jim Hansen reproduced an El Nino

[3] On presenting a copy of his work on celestial mechanics to Napoleon, the emperor remarked, "M. Laplace, they tell me you have written this large book on the system of the universe, and have never even mentioned its Creator." Laplace replied, "I had no need of that hypothesis."

[4] The Bjerknes circulation theorem generalized results of Helmholtz to atmospheric flows. The statement of the theorem was "A closed curve in a fluid possess, in its circular motion, an acceleration which is equal to the number of isobaric-isoteric solenoids it contains." The statement was based on an elegant proof of the equation $\frac{DC}{Dt} = \int\int \mathbf{B} \times \mathbf{G} \cdot dA$, where $\mathbf{B} = \nabla\alpha$, $\mathbf{G} = -\nabla p$, and $\alpha = 1/\rho$.

[5] Clay Mathematics Institute: http://www.claymath.org/millennium/.

(ENSO) signal using a global coupled climate model. Apparently, this oscillation as well as others such as the Pacific Decadal Oscillation (PDO) are modes of the dynamics coupling oceans and atmospheres. The ability to predict several seasons in advance that these oscillations are beginning or ending has raised the stakes in the whole climate and weather research effort. Already agricultural planning has absorbed the skill of these models to save millions of dollars in lost crops. And what is hoped, but so far not demonstrated, is that particular episodes of drought and severe weather can be explained through these physical and mathematical properties of the dynamical system. The theory of these climate oscillations, though not complete, has gone a long way toward explaining the natural variability of the climate system. This puts in perspective both quantitatively and qualitatively the other climate forcings, e.g., solar fluctuations, volcanic eruptions, and greenhouse gas loading.

1.4 ▪ The Advent of Climate Modeling

We have come to view the climate as an (attractive) state of an earth system that includes ocean, atmosphere, land, ice, and ecological and human components. This grand unification theory is still in the early stages of scientific investigation with key, long term observational data and more sophistication in the modeling needed to put all the facts into their correct contexts. Frances Bretherton may have been the first person to sketch the wiring diagram of how all the major pieces are connected. Syukuro Manabe produced the first modern climate model in 1967, predicting that if the earth's atmospheric concentration of carbon dioxide, CO_2, were doubled, the globally averaged surface temperature should rise by $2.3^o C$.[6] The discovery from this model was that if water vapor and moist processes are included, then the global surface temperature increase is amplified compared to that of a dry atmosphere. When earth system theory is mature, we have the hope of telling what is and is not caused by global warming.

Emissions of CO_2 into the atmosphere from fossil fuel burning from the industrial revolution onward is the cause. How the system will change as we continue to discharge our energy waste is the subject of current research, but the scientific community collects its best estimate every six years or so in the form of the Assessment Reports of the U. N. Intergovernmental Panel on Climate Change (IPCC).

The general circulation of the atmosphere and oceans is the granddaddy of all fluid flow problems, so until the dynamics of the flow are correctly understood and solved, the field will not be mature. Aggressive computational science and parallel computer projects have characterized the last decade in an attempt to generate algorithms and methods that solve the flow problem [46, 47]. But a challenge remains with only some evidence that the gross features of the flow are being solved with enough accuracy. For example, at a certain resolution or grid spacing in a discrete approximation to the mathematical equations, the baroclinic instabilities and ocean eddies are reproduced in simulations with convincing fidelity. The parallel computing technology continues on a path to definitively solve the flow problem by shear brute force of computation. Algorithm research that allows high performance computers to effectively simulate weather and climate has also seen a steady advance. Warren Washington and Akira Kasahara were among the first to bring a full climate model to fruition [14, p. 18]. The advanced computing power allowed general circulation models of the atmosphere and the ocean to be coupled together in the late 1980s. At first this was done with a procedure called "flux correction" that constrained

[6]The fundamental connection between warming and CO_2 concentration, known as the greenhouse effect, goes back to Fourier (1768–1830) and Arrhenius (1859–1927). Attention was renewed in 1956 when M. Budyko published *Heat Balance of the Earth's Surface*.

the ocean in order to maintain the present climate state [14, p.154]. But improved ocean models with more resolution as well as better representation of clouds in the atmospheric models allowed this constraint to be dropped. From an engineering standpoint, the solutions we are able to generate are good enough to produce significant and useful results. Understanding the interaction of the components of the climate system is dependent on this research direction and allows us to quantify climate change as distinct from natural climate variability. Model results form a key part of the IPCC assessment and the projections of climate change.

1.5 ▪ Entering the Global Warming Debate

The problem being addressed in the 1970–2000 period was whether or not climate change was occurring and whether humans were causing it. Warren Washington has written a personal memoir [44] about this period which offers insight into the policy discussions at the highest levels of government. After the IPCC Fourth Assessment Report in 2007 that spread Nobel Prizes throughout the climate science community, a qualitative shift occurred. Models are now being used to bound the consequences of climate change and plan mitigation strategies. The consequences of climate change, otherwise known as impacts, require much more comprehensive and detailed information from the models. In particular, the models are being used in high resolution configurations to develop regional and local projections and, often in conjunction with downscaling using mesoscale weather models, extreme weather statistics. The impact of heat waves, droughts, and flooding is arguably of first order in anyone's prioritization of the issues associated with a changing climate.

The prognosis from the models [18] is grim even if somewhat conservative, based on optimism of how emissions could be controlled. The current path for emissions traces a curve that exceeds everyone's expectation. Hence we may expect the increases in global mean temperature to be even higher than expected. By the end of the century (2100) an increase of 3–5^oC degrees may be nudged upward to 4–6^oC. This being said, we still do not have any confidence in predicting that the next decade will be warmer or colder. And this leads to the discussion of the difference between mitigation and adaptation to climate change.

The big question for mitigation is what we are going to do with all this carbon. First, how quickly can we shut off the carbon waste spigot, and, second, how will we balance carbon in a global budget that includes energy production, managed ecosystems for bioenergy, and economic growth for human prosperity and flourishing. The earth system must be understood to include the carbon cycle as well as other chemical components that affect the earth's energy balance and ecological processes [14, p. 177]. Simulation and modeling, backed by careful theory, extensive measurements, and observations, are the backbone for the science that will tell us when our efforts are enough. Technological innovations in solar energy, bioenergy, and nuclear energy on the supply side will need to be met with aggressive conservation measures on the demand side. The politics of how this could or should happen are the responsibility of the citizenry, and I am no better prepared than anyone else to discuss this. The problem is that we are seeking a compromise between our present habits and our long term well being. Mitigation is the problem of defining this long term goal and the way to get there.

Chapter 2

The Fundamental Theorem of Fluid Flow

I hope you remember the Fundamental Theorem of Calculus that relates integrals and derivatives for differentiable functions on the real line. The multidimensional version of the Fundamental Theorem is the Divergence Theorem that relates the integral of the divergence of a vector field to an integral around the boundary,

$$\int_{\Omega} \nabla \cdot \mathbf{v} dV = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} dA. \tag{2.1}$$

In this the normal vector $\mathbf{n}$ is *always* taken in the exterior direction to the region $\Omega \subset \mathbb{R}^n$. The Fundamental Theorem of Fluid Flow also relates derivatives and integrals but for functions carried on a flow. The material derivative will be the change in the function as observed by one riding on a particular particle moving with the flow.

2.1 ▪ Flow Maps and the Fundamental Theorem

The mathematical idea of a flow is a mapping of points from one location in three space to another. That is, if you are given an initial location of a point in space, $\mathbf{X}$, then at a time t the new location will be specified by $\mathbf{x}(t) = \Phi(\mathbf{X}, t)$. Expressed in terms of components this can be viewed as a coordinate transformation, a mapping of the region to itself, with variables $x = \xi(\mathbf{X}, t), y = \eta(\mathbf{X}, t)$, and $z = \zeta(\mathbf{X}, t)$. We will assume that this flow function is well defined and invertible so that trajectories do not cross and give us crazy situations like two particles occupying the same position at the same time but headed in different directions. This will be the case if the function is smooth enough to be differentiated in whatever manner we please.

The Fundamental Theorem can be stated as follows.

Theorem 2.1.1 (FTFF). *Let $\Phi(\mathbf{x}, t)$ be a flow and*

$$J(\mathbf{x}, t) = \left| \frac{\partial \Phi}{\partial \mathbf{x}} \right|; \tag{2.2}$$

then

$$\frac{dJ}{dt} = J(\mathbf{x}, t) \nabla \cdot \mathbf{v}, \tag{2.3}$$

where $\frac{dJ}{dt}$ is the total (or material) derivative and $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ with $\mathbf{x}(0) = \mathbf{X}$.

Proof. The velocity of the fluid is defined in terms of the flow by the equation

$$\frac{\partial \Phi}{\partial t} = \mathbf{v}(\Phi(\mathbf{x},t),t) \text{ with } \Phi(\mathbf{x},0) = \mathbf{X}. \tag{2.4}$$

Holding $\mathbf{x}$ fixed, we differentiate the Jacobian of the (coordinate/flow) transformation

$$J = det\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} \tag{2.5}$$

as

$$\frac{dJ}{dt} = det\begin{pmatrix} \frac{d}{dt}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{d}{dt}\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{d}{dt}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + det\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{d}{dt}\frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{d}{dt}\frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{d}{dt}\frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + det\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{d}{dt}\frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{d}{dt}\frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{d}{dt}\frac{\partial \zeta}{\partial z} \end{pmatrix}. \tag{2.6}$$

Since $\frac{d\xi}{dt} = u$, $\frac{d\eta}{dt} = v$, and $\frac{d\zeta}{dt} = w$,

$$\frac{dJ}{dt} = det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + det\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial \zeta}{\partial z} \end{pmatrix} + det\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{2.7}$$

Further, since $u(\Phi(\mathbf{x},t)) = u(\xi(\mathbf{x},t), \eta(\mathbf{x},t), \zeta(\mathbf{x},t), t)$,

$$\frac{\partial u(t,\Phi(\mathbf{x},t))}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial \zeta}{\partial x}, \tag{2.8}$$

and so forth, so substituting into (2.7),

$$\frac{dJ}{dt} = \frac{\partial u}{\partial x}J + \frac{\partial v}{\partial y}J + \frac{\partial w}{\partial z}J = J\nabla\cdot\mathbf{v}. \qquad \square \tag{2.9}$$

2.2 ▪ The Transport Theorem

The reason this is so fundamental is that it is used to prove the transport theorem, which is used to derive the equations of motion for fluid flow, and informs the notion of the material derivative,

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f. \tag{2.10}$$

Theorem 2.2.1 (Reynolds transport). *For any function $f(\mathbf{x},t)$ and a volume $\Omega_t = \Phi(\Omega_0, t)$ following the flow,*

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \int_{\Omega_t}\left(\frac{df}{dt} + f\nabla\cdot\mathbf{v}\right)dV. \tag{2.11}$$

Proof.

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \frac{d}{dt}\int_{\Omega_0} f(t,\Phi(\mathbf{X},t))J\,dV_0. \tag{2.12}$$

Since the integral involves time only in the integrand,

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \int_{\Omega_0}\left(J\frac{df}{dt} + f\frac{dJ}{dt}\right)dV_0 = \int_{\Omega_0}\left(\frac{df}{dt} + f\nabla\cdot\mathbf{v}\right)J\,dV_0. \tag{2.13}$$

On transformation back to the moving volume Ω_t, the result follows. □

As a special case, if $f \equiv 1$, then

$$\frac{d}{dt}\int_{\Omega_t} dV = \int_{\Omega_0}\frac{dJ}{dt}dV_0 = \int_{\Omega_0}\nabla\cdot\mathbf{v}J\,dV = \int_{\Omega_t}\nabla\cdot\mathbf{v}\,dV. \tag{2.14}$$

If we make the assumption that the fluid is incompressible[7] and volumes following the flow have constant volume, then the Jacobian is not changing, and we must have $\nabla\cdot\mathbf{v} = 0$ everywhere in the flow. The equation also has some other variants:

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \int_{\Omega_t}\left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{v})\right)dV \tag{2.15}$$

and

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \frac{\partial}{\partial t}\int_{\Omega_t} f + \int_{\partial\Omega_t} f\mathbf{v}\cdot\mathbf{n}\,dA. \tag{2.16}$$

In this last expression the Divergence Theorem has been applied to transform the volume integral into an area integral around the boundary of the moving volume. If the continuity equation, (2.19), holds for ρ, then the following also holds for arbitrary f:

$$\frac{d}{dt}\int_{\Omega_t}\rho f\,dV = \int_{\Omega_t}\rho\frac{df}{dt}dV. \tag{2.17}$$

2.3 ▪ Fluid Flow Conservation Laws

The transport theorem is used to derive the conservation equations for mass, momentum, and energy with $f = (\rho, \rho\mathbf{v}, \rho e)$. It is the most elegant way of deriving the equations of motion.

Let $f = \rho$, the density. Then conservation of mass would require that "the mass of fluid in a material volume Ω_t does not change as it moves with the fluid." Or, simply,

$$0 = \frac{d}{dt}\int_{\Omega_t}\rho\,dV = \int_{\Omega_t}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right)dV. \tag{2.18}$$

Since the volume is arbitrary we can conclude that the integrand is zero everywhere and the equation for mass conservation is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{2.19}$$

Let $f = \rho\mathbf{v}$ be the momentum. Then conservation of momentum states "the rate of change of linear momentum of a material volume equals the resultant force on the volume."

$$\frac{d}{dt}\int_{\Omega_t}\rho\mathbf{v}\,dV = \int_{\Omega_t}\rho\frac{d\mathbf{v}}{dt}dV = \int_{\Omega_t}\rho F + \int_{\partial\Omega_t}\mathbf{t}\cdot\mathbf{n}\,dA. \tag{2.20}$$

[7]If the density is not constant but $\frac{d\rho}{dt} = 0$, as in a stratified fluid, then the flow may be incompressible while the fluid is not. The flow will still be divergence-free. You can see this by considering the continuity equation (2.19).

For a perfect fluid, the stress tensor is simply defined by pressure[8] as $\mathbf{t} = -p\mathbf{n}$. The equation reduces to the pointwise statement

$$\rho \frac{d\mathbf{v}}{dt} = \rho F - \nabla p. \tag{2.21}$$

In the derivation we have used that ρ satisfies the mass conservation (continuity) equation.[9]

For the energy equation,[10] let $f = \rho e$. The conservation of energy statement is "the rate of change of the total energy is equal to the rate of internal generation of heat plus the rate at which work is done on the volume plus the rate at which heat is conducted into the volume." Using a perfect fluid with $\mathbf{t} = -p\mathbf{n}$, and applying the transport theorem,

$$\frac{d}{dt}\int_{\Omega_t} \rho e\, dV = \int_{\Omega_t} \rho \frac{de}{dt} dV = \int_{\Omega_t} \rho Q + \int_{\partial\Omega_t} \mathbf{t}\cdot\mathbf{v}\, dA - \int_{\partial\Omega_t} \mathbf{q}\cdot\mathbf{n}\, dA, \tag{2.22}$$

where Q is a volume heating source and $\mathbf{q}$ is the heat flux. For a perfect fluid we complete the Euler equations as

$$\rho \frac{de}{dt} + p\nabla\cdot\mathbf{v} = \rho Q - \nabla\cdot\mathbf{q}. \tag{2.23}$$

2.4 ▪ Relation to Semi-Lagrangian Methods

Since the idea of the flow is generally not directly exposed, I'd like to linger here for a little and consider the relation to numerical methods since semi-Lagrangian methods rely on particle tracking the flow. We will explore a direction based on an idea Paul Fisher suggested,[11] a sort of extension of the Courant–Isaacson–Reses method of upwinding and a new way to configure semi-Lagrangian transport methods.

The idea of a semi-Lagrangian method is very simple and can be thought of as one of the oldest methods for weather forecasting. You simply need to know where the weather is coming from (a departure point) and when it will be arriving (at the arrival point). Saying that the weather will blow in from departure point to arrival point is the same as an approximation of the material derivative as

$$\frac{df}{dt} = \frac{f(t^{n+1}, \mathbf{x}_a) - f(t^n, \mathbf{x}_d)}{t^{n+1} - t^n} = 0. \tag{2.24}$$

Or, equivalently, $f(t^{n+1}, \mathbf{x}_a) = f(t^n, \mathbf{x}_d)$, a simple assignment.

Suppose we treat the coordinates $\mathbf{x}$ as advected scalar quantities? Moving from a time level t^n to the new time level t^{n+1} with velocity $\mathbf{v}$, the value of $\mathbf{x}$ at the arrival point $\mathbf{x}_a$ is the departure point $\mathbf{x}_d$. If you want this to be very accurate, conserving areas, etc.,

[8] By taking a different constitutive relation for the stress tensor one obtains the Navier–Stokes equations. For example, for an incompressible fluid the stress tensor is $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$ with $\mathbf{t} = \mathbf{n}\cdot\mathbf{T}$. The deformation tensor is defined by $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$, with the i, j subscript indicating differentiation of the ith component in the j direction. The stress tensor then has the interpretation that $T_{i,j}$ is the jth component of force on a surface element with exterior normal in the ith direction.

[9] In the incompressible Navier–Stokes case an additional term is added from the stress tensor, $\nabla\cdot(2\mu\mathbf{D})$. Without going into the calculus of tensors this reduces to $\mu\nabla^2\mathbf{v}$.

[10] For total energy, let $f = \rho(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \Phi)$, the total energy which is the sum of the internal, kinetic, and potential energy.

[11] Personal communication.

and not creating artificial divergence, then the numerical method for advecting must be carefully considered.

Consider, for argument, the unusual choice of a global spectral algorithm for computing the flow. In order for the FTFF to hold in a numerical sense, the accurate representation of J is important. A spectral representation of Φ would give excellent accuracy for the partial derivatives of Φ used in the definition of J, so this might be a good choice. If you are using a spectral representation, then presumably $\nabla\cdot$ and $\nabla\times$ are well approximated for other fields as well. Representing Φ in spectral space is like an isoparametric finite element method where the geometry is interpolated using the same shape functions as the solution.

Let $\Phi(t,\mathbf{x}) = \sum_{m,n} \xi_n^m(t) Y_n^m(x)$ be the spectral expansion of the flow. For spherical flows, the Y_n^m are the spherical harmonic basis functions. To avoid the discontinuities of the velocity at the poles, we will use Cartesian coordinates and velocities in the spherical case, though we could work around this in spherical coordinates by using the vector spherical harmonics [39]. This global solution for Φ will replace the individual particle tracks of the semi-Lagrangian algorithm.

A global Galerkin method applied to (2.4) gives

$$\dot{\xi}_n^m = \int_{S^2} v(t,\Phi(t,\mathbf{x}))Y_n^m(\mathbf{x})dx = \int_{S^2} v(t,\mathbf{x})Y_n^m(\mathbf{x})J(t,\mathbf{x})d\mathbf{x}. \qquad (2.25)$$

Now if J can be evaluated accurately, then the integration should be very accurate. But this can be done from the FTFF and doesn't require anything but a time integration of (2.3). The algorithm is as follows:

1. Given $\mathbf{v}(t,\mathbf{x})$ compute $\nabla\cdot\mathbf{v}$ at the Gaussian grid points of a spectral approximation.
2. Update J at the grid points using (2.3).
3. Evaluate $\langle \mathbf{v}J, Y_n^m\rangle$ for each (m,n).
4. Integrate $\dot{\xi}_n^m$ to update the spectral coefficients of Φ.
5. Use spectral synthesis to get Φ at grid points.

These new grid point values are the updated advected coordinates and give the departure point coordinates for each grid-arrival point. This is done with spectral accuracy modulo the accuracy of the ODE time advancement algorithms.

Now there are a few things to note. If the divergence of the velocity is nonzero as in a compressible flow, then $0 < J < \infty$ will not remain constant and indeed can grow or diminish exponentially fast where there are persistent sources or sinks. This is troublesome for numerical methods that introduce grid approximation errors at special points such as the poles, icosahedral points, or interfaces.

Once the departure points are calculated, a semi-Lagrangian method must interpolate the advected quantity from the grid points to the irregular departure points. This introduces an error, potentially destroying any accuracy maintained in the computation of the departure point locations. The commensurate interpolation is a spectral synthesis at the irregular points. But this is known to have problems with ringing and Gibbs oscillations; it does not give a monotone advection scheme. This subject will be taken up in a lecture on Godunov schemes and clearly shows the contrast between accuracy of a Courant–Fredrick–Rheses method and conservative methods.

Chapter 3

Singular Sturm–Liouville Problems and the VSE

This is a topic that may not have been covered in your course on ordinary differential equations (ODEs) when you studied boundary value problems. More than likely, if Sturm–Liouville problems were mentioned, the theory for regular (nonsingular) problems was presented. So this lecture will explore some of the additional complications of the singular problem, and for this effort we will be rewarded with some fundamental tools for approximation of functions. Since numerical analysis is almost always applied mathematics and approximation is the whole point of it, I'd like to suggest that this subject ranks, on the graduate level of numerical analysis, with how Taylor's theorem ranks on the undergraduate level. That is to say, the functional framework of orthogonal polynomials is pervasive in the same manner as the pointwise approximation of a Taylor expansion.

3.1 ▪ Existence and Uniqueness Theorem

Definition. A Sturm–Liouville problem

$$-(p(x)u_x)_x + q(x)u = \lambda w(x)u \quad \text{on } I = [a,b] \tag{3.1}$$

is *regular* if $p, w > 0$ on the finite interval I. If either p or w is zero on I or if the interval is infinite, then the problem is *singular*. The appropriate boundary conditions that complete the description of the problem will be discussed later.

The theory says [11] that there are a discrete set of positive (real) eigenvalues $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$, and a set of orthogonal eigenfunctions $\psi_1, \psi_2, \dots$ associated with the eigenvalues, where orthogonality is defined by the function space inner product

$$\langle f, g \rangle = \int_I f(x)g(x)w(x)dx. \tag{3.2}$$

That is, the eigenfunctions form a natural basis for the Hilbert space of functions $H = L^2(I)$ with the w-weighted inner product. The proof must consider the regular problem first and then the singular problem. The complications come from the details of boundary conditions. The proof of the theorem uses the self-adjoint property of the operator

$$L(u) = -(p(x)u_x)_x + q(x)u. \tag{3.3}$$

With proper satisfaction of boundary conditions restricting the space H on which the operator is applied, an operator is self-adjoint if

$$\langle Lu, v\rangle = \langle u, Lv\rangle \quad \text{for all } u, v \in H. \tag{3.4}$$

The Sturm–Liouville operator is self-adjoint because

$$\int_I (-(pu_x)_x + qu)v\,dx = -p(x)u_x v|_a^b + \int_I pu_x v_x\,dx + \int_I quv\,dx \tag{3.5}$$

$$= -(pu_x v - pv_x u)|_a^b + \int_I (-(pv_x)_x + qv)u\,dx, \tag{3.6}$$

and if and only if the boundary term drops for all $u, v \in H$, then L is self-adjoint. Using this, suppose that λ is an eigenvalue and ψ is an eigenfunction. Since these are complex we will note that for complex functions the inner product $\langle u, v\rangle = \int u\bar{v}$, so that

$$\langle L\psi, \psi\rangle - \langle \psi, L\psi\rangle = (\lambda - \bar{\lambda})\langle \psi, \psi\rangle = 0. \tag{3.7}$$

This can only be if λ is real since $||\psi||^2 = \langle \psi, \psi\rangle \neq 0$. Now suppose that λ_1 and λ_2 are two distinct eigenvalues with eigenfunctions ψ_1 and ψ_2. Since L is self-adjoint,

$$0 = \langle L\psi_1, \psi_2\rangle - \langle \psi_1, L\psi_2\rangle = (\lambda_1 - \bar{\lambda}_2)\langle \psi_1, \psi_2\rangle. \tag{3.8}$$

So $\langle \psi_1, \psi_2\rangle = 0$ and the two eigenfunctions are orthogonal. Note that eigenfunctions are defined on H and satisfy homogeneous boundary conditions so that the integration by parts boundary term drops.

3.2 ▪ Orthogonal Polynomial Expansions

Many of the remarkable polynomial expansions in approximation theory derive from this equation and theorem. For example, consider the interval $I = [-1, 1]$ and the Sturm–Liouville problem with $p(x) = 1 - x^2$, $q = 0$, and $w = 1$. Since $p(-1) = p)1) = 0$, the Sturm–Liouville problem is singular. The eigenvalues are $\lambda_n = n(n+1)$, integers(!), a remarkable thing in itself, and the eigenfunctions are the Legendre polynomials, 1, x, $\frac{1}{2}(3x^2 - 1)$, $\frac{1}{2}(5x^3 - 3x)$, etc.

Now any useful approximation is finite. Since the functions form a basis, one can simply truncate and form a finite dimensional subspace of the infinite dimensional $L^2(I)$. Once you have this finite dimensional space you can use it to approximate the whole space. So for a quadrature rule replacing the continuous integrals, if

$$f(x) \approx \sum_{i=1}^{N} a_i \phi_i(x), \tag{3.9}$$

then

$$\int f(x)\,dx \approx \sum_{i=1}^{N} a_i \int \phi_i(x)\,dx. \tag{3.10}$$

If interpolation is the name of the game, then $f(x^*) \approx \sum_{i=1}^{N} a_i \phi_i(x^*)$. Basically, any linear operator $L : L^2(I) \to H$ on the infinite dimensional space can be approximated on the finite dimensional space using

$$L_N f = \sum_{i=1}^{N} a_i L\phi_i. \tag{3.11}$$

If you are trying to find f that satisfies $Lf = g$, then the discrete version solves for the a_i's such that

$$\mathbf{L}_N \mathbf{a} = \mathbf{g}, \tag{3.12}$$

where

$$L_N^{i,j} = \langle L\phi_j, \phi_i \rangle \text{ and } g_i = \langle g, \phi_i \rangle. \tag{3.13}$$

For the Sturm–Liouville operator, the discrete operator would be

$$L_N^{i,j} = \int_I \left[((p\phi_j)_x)_x + q\phi_j \right] \phi_i. \tag{3.14}$$

3.3 ▪ Spectral Element Discretization for Sturm–Liouville Problems

So what about numerically solving differential equations and boundary value problems such as the singular Sturm–Liouville problem? In the special case associated with the name Laguerre, we set $p(x) = xe^{-x}$, $q(x) = 0$, and $w(x) = e^{-x}$, where the interval is $[0, \infty)$. This is singular at both the left endpoint $x = 0$ and because of the infinite interval. The eigenfunctions are 1, $1-x$, $\frac{1}{2}(x^2 - 4x + 2)$, $\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$, etc. Shouldn't we be able to use these basis functions to solve differential equations on a semi-infinite interval? They are, after all, the natural basis for functions on this interval, at least of reasonable functions that don't blow up faster than an exponential as they go to infinity. Canuto [8, p.72] explains the options for solving the boundary value problem on the semi-infinite domain and maintaining spectral convergence of the numerical method:

- expand in Laguerre functions for a global spectral approximation,
- map the semi-infinite interval to the finite interval [-1,1] and expand in Chebyshev polynomials, and
- truncate the domain $[0, x_{max}]$ and use Chebyshev expansion.

To push this a little further, we develop a spectral element discretization [33, 5] of the singular Sturm–Liouville problem that allows for a variable level of spectral approximation within each element. We divide the interval I into n_e finite elements, $I = \cup_e I_e$. For the semi-infinite interval, the last element will extend to infinity. Rewrite (3.3) using a weak formulation and integration by parts as

$$\int_I (pu_x)v_x + quv\,dx = \int_I fv\,dx. \tag{3.15}$$

On an element by element basis this is replaced by

$$\sum_{e=1}^{n_e} \int_{I_e} (p(u_h^e)_x)(v)_x + qu_h^e v\,dx = \sum_{e=1}^{n_e} \int_{I_e} fv\,dx, \tag{3.16}$$

where $u_h^e = \sum_{e=1}^{n_e} \sum_i u_i^e \phi_i^e(x)$. The generic notation u_h is often used to denote an approximation dependent on some mesh size h. The $v \in \mathcal{V}$ are chosen in a test space so that the boundary terms vanish. For the semi-infinite interval, this boundary assumption is critical and may influence the modes of the solution.

Substituting the expansion u_h into each element and using the same ϕ_j^e basis for $\mathscr{V}$, we arrive at an element matrix equation for the unknown coefficients u_i^e:

$$A^e \mathbf{u}^e = F^e, \tag{3.17}$$

where

$$A_{ij}^e = \int_{I_e} \left[p(x)\left(\phi_i^e(x)\right)_x \left(\phi_j^e(x)\right)_x + q(x)\phi_i^e(x)\phi_j^e(x) \right] dx. \tag{3.18}$$

Since this applies on each element an assembly of the element matrices (from the sum in (3.16)), a full system matrix is required to incorporate all the unknowns common to elements. For example, if the endpoints of the intervals include nodal points where the basis is nonzero, as in a Cardinal basis [5], the unknowns at those nodal points involve more than one element and their equations must be summed. What results is a system matrix for all the unknowns $\mathbf{u}$,

$$A\mathbf{u} = F. \tag{3.19}$$

The eigenvalue problem can then be posed in a discrete form as

$$A\mathbf{u} = \lambda \mathbf{u}. \tag{3.20}$$

3.4 ▪ A Compact Discretization

Another interesting discretization that also yields spectral convergence is a compact method based on the Laguerre polynomials [26]. The approximation is based on discrete points of a grid $\{x_j\}$, and there are only a finite number of them since we cannot keep running out to infinity on a computer. A compact method for the derivative [34] means an approximation of the form

$$\sum_{j=L}^{M} A_{ij} u_x(x_j) = \sum_{j=J}^{K} B_{ij} u(x_j). \tag{3.21}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ will be determined to get an accurate approximation about the point x_i. In terms of a stencil, the formula will use several implicit derivative points and several function value points. These do not need to be on the same grid; for example, staggered grids can be used. The compact procedure works for more than a derivative. We could use it for interpolation or quadrature—actually, any linear operator. To make this generalization, write the unknown point values as $\mathbf{d}$ and the known point values as $\mathbf{p}$. The compact approximation seeks matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{Ad} - \mathbf{Bp} = 0$,

$$(\mathbf{D}, \mathbf{P}) \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = 0, \tag{3.22}$$

where $\mathbf{D}$ and $\mathbf{P}$ are Vandermonde matrices for $\mathbf{d}$ and $\mathbf{p}$.

I'd like to propose that we choose the method to be exact on the first few Laguerre polynomials. That is, we will have an exact approximation for the unknown for functions in a subspace of the larger space when evaluated at the grid points. The equation for the method coefficients evaluating a derivative ($\mathbf{d} = u_x$, $\mathbf{p} = u$) is

$$\begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ L_0'(x_L) & \dots & L_0'(x_M) & -L_0(x_J) & \dots & -L_0(x_K) \\ L_n'(x_L) & \dots & L_n'(x_M) & -L_n(x_J) & \dots & -L_n(x_K) \end{pmatrix} \begin{pmatrix} A_i^T \\ B_i^T \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.23}$$

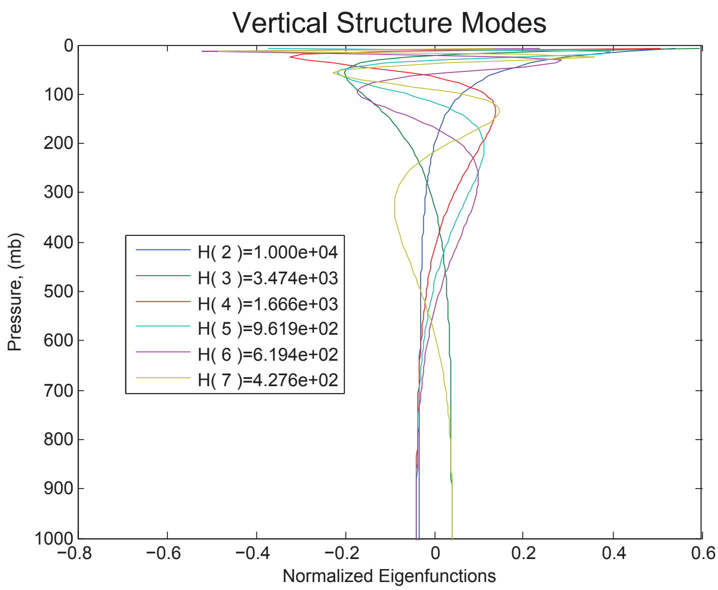

Figure 3.1. *The modes of the vertical structure equation as computed with a compact method. The oscillatory behavior of higher modes is evident as the upper ($p = 0$) boundary is approached.*

Suppose we let $\mathbf{d} = L(u)$, where L represents the self-adjoint linear operator of the Laguerre equation. Then the equation for the method coefficients is

$$\begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ \lambda_0 L_0(x_L) & \dots & \lambda_0 L_0(x_M) & -L_0(x_J) & \dots & -L_0(x_K) \\ \lambda_n L_n(x_L) & \dots & \lambda_n L_n(x_M) & -L_n(x_J) & \dots & -L_n(x_K) \end{pmatrix} \begin{pmatrix} A_i^T \\ B_i^T \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.24}$$

To solve the equation $Lu = f$ using the compact method, note that $\mathbf{A}^{-1}\mathbf{B}u = f$ on a pointwise basis, so that $u = \mathbf{B}^{-1}\mathbf{A}f$. There is some assembly required to deal with the implicit points solving on the whole grid.

3.5 ▪ Vertical Structure Equation

To see how this might work for an equation that doesn't have these nice eigenfunctions, we will try the vertical structure equation of the atmosphere. This is a singular Sturm–Liouville problem, with a zero coefficient at one endpoint. This can be related to a semi-infinite interval with a change of variable to height $z = -\ln x$, so we might expect the Laguerre polynomials and the associated weighted inner product to be of value in the approximation. The vertical structure equation (VSE) is [12]

$$(x^2 u_x)_x + qu = 0, \tag{3.25}$$

where the x-coordinate is pressure in the atmosphere and $q = \frac{RT}{gh}$ are constants of an isothermal atmosphere. The boundary conditions are $u_x + \frac{x}{x}u = 0$ at $x = 0, p_s$.

In the MATLAB exercises [15], the compact method is applied to the VSE, and the eigenvalues (modes) are computed. It is interesting that these modes also appear in a normal mode expansion of the hydrostatic (or nonhydrostatic [23]) equations of motion from a separation of variables. This means that the VSE modes are indicative of the way the atmosphere responds dynamically in the vertical. In developing the theory of geostrophic turbulence, Charney [9] added a vertical dimension assuming that the vertical structure could be approximated by a regular Sturm–Liouville problem. This restricts the modes of variability by using different normal modes than the problem actually calls for [49]. This "mistake" may have influenced the development of baroclinic atmospheric models, and few are aware of it today.

Any discretization of the vertical can be viewed as giving an approximation of the VSE modes. Depending on how that discretization is developed, it may or may not be true to the vertical operator. This has implications for the type of discretizations that are acceptable—adequate resolution and correct boundary conditions that should be applied in the vertical.

With the correct normal modes as a way to judge the resolution of vertical discretizations and the correct way of applying boundary conditions in the vertical, it is possible to distinguish between the various grid choices and discrete methods in the vertical. Sadly, most global circulation models make a point of pushing resolution in the horizontal without considering that matching resolution is needed in the vertical to capture the correct dynamical modes of the atmosphere.

Chapter 4

Numerical Solution of Conservation Laws

Theorem 4.0.1 (Godunov). *Linear, monotone implies first order accurate* [27].

The problem we are discussing is strikingly simple:

$$u_t + a u_x = 0 \text{ with initial condition } u(0,x) = u_0(x). \tag{4.1}$$

We want to generate a numerical solution based on a method that is linear (the problem is linear), preserves a known property of the solution near discontinuities (monotone), and is high-order accurate. The Godunov theorem says it cannot be done.

We know that the solution obeys the set of equations

$$\frac{dx}{dt} = a, \text{ so, up to a constant } x = at + c, \tag{4.2}$$

$$\frac{du}{dt} = u_t + u_x \frac{dx}{dt} = u_t + a u_x = 0. \tag{4.3}$$

So $u(at + c) = u_o(c)$ for each c and for all t is the exact solution. This is simply a moving profile translating with speed a. One can see that if the initial data u_0 is monotone (increasing or deceasing), then so is $u(t^*, x)$ for any t^*. Numerically, this property of monotonicity preserving through time is that if $u_0(x_j) \geq u_0(x_{j+1})$ for all j, then $u^n(x_j) \geq u^n(x_{j+1})$ for all j. This property prevents oscillations from forming near discontinuities.

The variation of numerical solutions is important because a small error may grow into a larger error. And if you don't know better, you might think some fine scale behavior is a result of physics rather than a numerical property of the method. On the other hand, we all have a certain faith and reliance in Taylor's theorem, which says reasonable functions can be expanded in power series with infinite accuracy.

A stronger condition than monotonicity preserving is that of a monotone method. If we have two intial conditions u_0 and v_0 with $v_0(x) \geq u_0(x)$ for all x, then we should have $v(t,x) \geq u(t,x)$ for all t. That a monotone numerical method will maintain this in the discrete sense of $V_j^n \geq U_j^n$ for all j implies that $V_j^{n+1} \geq U_j^{n+1}$ for all j.

It is not much of a stretch to see that monotone implies monotonicity preserving.

4.1 ▪ Dilemma of the Godunov Theorem

So, what to choose? Pick *two* of the following: high order, monotone, or linear! The logical implications of the choices are as follows:

(a) high order: monotone implies nonlinear;

(b) linear: high order implies not monotone; or

(c) linear: monotone implies first order.

At this point you have to know something about the problem you are solving, and this often leaves the numerical analyst at a loss. What is most important for the problem? It helps to think about the physics.

The easiest thing to do is (c) and sacrifice the accuracy. There is nothing wrong with this choice, but you cannot publish papers about it, and one is left with the nagging suspicion that you could do better. In fact, the error of a first order scheme may appear as a phantom diffusion term (νu_{xx}) which is decidedly not the physics of advection. You can see this by examining the truncated terms of the Taylor expansion.

The choice (b) is also fairly easy and in particular circumstances may be a good choice, for example, when the solutions of interest are very smooth and there are no complications from wiggles or propagation of noisy signals. A filter to remove the wiggles or other nonphysical features may work well, though these fixers and projections seem arbitrary. If there is a source term instead of zero on the right-hand side, it may negate or overwhelm any wiggles the high order linear scheme produces, and this would be a lucky situation in which the numerical method is let off the hook. On the other hand, a solution may not permit wiggles. For example, if the source term represents a chemical reaction and the solution u is the concentration of a chemical species, it will not do for u to become negative. Reactions just don't work with negative concentrations.

Generally a conservation law conserves more than one property. For example, the nonlinear Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{4.4}$$

conserves not only u but also its square u^2. So a numerical method that preserves as many of the equation's invariates as possible is an added bonus.

This leaves us with (a), i.e., nonlinear schemes. This revolting development has led to some brilliant ideas and ways to cheat on monotonicity to make things easier. The development of upwind, TVD, and ENO schemes is another chapter. For now we will return to the Godunov method, which is often good enough for conservation laws.

4.2 ▪ The Godunov Method

Let the mesh be divided up into small control volumes, with $x_j \in [x_{j-1/2}, x_{j+1/2}]$. Then a control volume average solution is denoted by

$$U_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)dx, \tag{4.5}$$

where $\Delta x = x_{j+1/2} - x_{j-1/2}$. The value within the control volume is constant when thought of as a function of space as it represents an average over the control volume.

Integrating the conservation equation $u_t + (f(u))_x = 0$ over the control volume and the time interval $[t^n, t^{n+1}]$,

$$\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} u_t\,dx\,dt + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} (f(u))_x\,dx\,dt \tag{4.6}$$
$$= U_j^{n+1} - U_j^n + \frac{\Delta t}{\Delta x}\left[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)\right].$$

The deviling details enter in the evaluation of the numerical flux,

$$F(U_j^n, U_{j+1}^n) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u^n(x_{j+1/2}, t))dt. \tag{4.7}$$

Each control volume can be thought of as a mini linear advection problem where the time evolution at the cell edge point $x_{j+1/2}$ is a constant control volume average U^n coming from either the right or the left depending on which way the wind blows. Because of the nonlinearity of the Burgers equation and conservation laws in general, it is a little more complicated than the linear advection case would indicate so a more refined approach considers the Riemann problem with rarefaction waves, etc., as the exact solution to the nonlinear problem over the small time interval. But as long as you take a small timestep, determined by the Courant number $C = |\frac{\Delta t}{\Delta x} U_j^n| \leq 1$, the value along the edge will remain constant, u^*, and the flux integral is simple to evaluate as

$$F(U_j^n, U_{j+1}^n) = f(u^*(U_j^n, U_{j+1}^n)). \tag{4.8}$$

What is nonlinear about this? It is the choice of the constant u^* that is nonlinear. If we always took this as the average of the U_j^n and U_{j+1}^n values, then it would be a linear scheme. But since we choose by solving the local Riemann problem and taking into account the direction of the wind, it is nonlinear. There are other important things to note. The Godunov method is exactly conservative because the discrete equation is written in flux form. The integral of U leads to a canceling out of all the flux terms so that $\int U^n dx = \int U^{n+1} dx$. Further, the scheme is monotone because the flux is computed from a constant u^* that is equal (in the upwind variant) to one or the other of U_j^n, U_{j+1}^n. But what about u^2? The Godunov method doesn't do so well here.

4.3 ▪ Numerical Conservation in the Atmosphere

For the conservation laws governing the flows of the atmosphere, one needs to take a more holistic view and make choices about which conserved quantities are most important to preserve. One proposal [40] makes a distinction between the idealized invariants of adiabatic frictionless flow and invariants that transfer the quantity from large scales to small scales through a dynamic cascade. These later must be approximated and damped in the fine scale as a means of closing the method, and this implies nonconservation at the level of accuracy of the method.

The quantities that are conserved by the adiabatic frictionless flow are mass, angular momentum, energy, potential temperature, potential vorticity, and tracers. Each conservation law that can be enforced reduces by one the dimension of the manifold that the solution must follow, and if the budget of the quantity is of interest scientifically, e.g., carbon, then there may be significant benefit regardless of the dynamical fidelity. Some conserved quantities like energy and potential enstrophy may be important for the numerical stability of methods as well. This speaks to the theoretical basis of any method as converging to a solution as time and space are increasingly resolved. Unfortunately, the mathematical knowledge at this time is not complete, and there is no proof that the compressible Euler equations have a unique solution for finite times. So, "convergence to which solution" may be an appropriate question. Do you like the one that stays bounded, but otherwise shows large oscillations, or do you like very smooth, damped solutions? In light of this situation, atmospheric modelers turn to the real atmosphere as the ultimate determinant of the kind of solution they are interested in.

One important observation about the atmosphere is that the kinetic energy spectra follows a k^{-3} power law for synoptic scales of motion and then transitions to a $k^{-5/3}$ law on scales of 100km or less. In this it looks much like two-dimensional turbulence theory would suggest on the large scales and more like three-dimensional turbulence on the fine scales. How and why the transition occurs at these scales is an open question. The resolutions of most climate and weather models are below this fine scale and only able to resolve synoptic scales of motion. A numerical method should try to duplicate not only the observed distributions of fields and their scale structures but also the observed sources and sinks. "The errors that arise from imperfect conservation in a dynamical core will depend on the magnitude of the imperfection in conservation compared to the true physical sources and sinks [40]." The spurious sources over the period of time you are interested in should drive the choices of conserved quantities. Thus, for climate simulations, the conservation of mass and energy is of great importance as loss of mass affects pressure directly, a sort of letting the air out of the balloon. For numerical weather prediction (NWP), the enstrophy and potential vorticity have sink time scales on the order of 10 days, comparable to eddy turnover times. Angular momentum is not one of the adiabatic frictionless conserved quantities, because it is exchanged across the lower boundary of the system. However, its conservation is important for the strength of the zonal wind and the meridional circulation. Many of the atmosphere's internal modes of variability link with this budget on the seasonal and interannual time scales. So again, thinking about the physics of the problem you are studying can influence the choices for a dynamical core.

One might naively assume that if we were just a little smarter, we would figure out how to conserve all the quantities and do it with high-order accurate methods. Indeed, one line of reasoning asserts that by increasing resolution the accuracy eventually pins the invariants. As a brief reminder of the actual situation, consider the entropy, $C_p \ln \theta$. This is just one of the infinite set of conserved quantities for adiabatic frictionless flow of the form $\rho f(\theta)$. The diabatic heating of the atmosphere is much larger than numerical sources/sinks that would be fixed by preserving a large quantity of the entropy invariants, so perhaps improvements in this aspect of the dynamics should be weighed with biases in the heating source terms. But the infinite invariants remind us how close the dynamical formulation is to Hamiltonian dynamical systems. One can also think of a dynamical method as a being generated from a nearby Lagrangian with its set of invariants derived from symmetries and Casimirs. Many people are beginning to look at approximations developed in this framework [35].

Chapter 5
The Evening News

The gradient of the geopotential and why it matters for understanding the evening news.

In this lecture we will cover some mathematical material dealing with the equations of motion of the atmosphere. Looked at another way, we will try to understand a weather map, and starting from that point of view, it has always seemed puzzling that the wind moves clockwise around a high pressure and counterclockwise around a low pressure. A bit of international intrigue enters when you find out that exactly the opposite holds in Chile. If you have been lucky enough to take vector calculus and learn Newton's laws of motion, then you may already know why the weather behaves the way it does. If not, then you will have to take some things for granted.

5.1 ▪ Getting Oriented

The first thing we take for granted is that the gradient of a function is a vector, denoted $\nabla\Phi$, where the function Φ depends on two or more space dimensions. And this vector points in a direction that is toward the steepest ascent. If you were climbing a mountain (Φ) with a 50 lb pack on, this would be the worst direction to go. A somewhat minor point, but still important is that the magnitude (length) of the gradient vector is how steep the slope is. So while you might enjoy a little gradient, you likely will become winded in the presence of a large gradient.

The second thing we take for granted is that up and down are very relative terms. You may think you know all about this, but almost anything involving global or international politics depends on your perspective. Let's start with a basic example of hanging a plumb line. Which way does it point? Down, right? Well...OK. Toward the center of the earth, right? But doesn't Newton's law of gravitation state that objects are attracted to the center of mass? It must be so! Gravity points down! Well, that is OK if that is your idea of down. We will denote by **k** the direction of vertical away from the center of the earth.

Let's suppose that the gravitational field can be represented by the gradient of a function. We will call this function the geopotential.[12] This gradient will point down (according to articles of faith one and two), and the strength of the gravitational field will be the length of the vector. So a point in equilibrium, like a plumb line after it stops swinging, will balance the forces acting on it. Since, in a glass of water, the water surface settles perpendicular to this plumb line, it is not hard to see that an ocean surface, ignoring tides

[12] It doesn't seem like much of an assumption, but this simple mathematical move has the mark of genius and brings potential theory into use for the description of weather and climate.

and waves, should rest on a geopotential surface, say, $\Phi = \Phi_0$. You know that sea level is not on a fixed radius: there is a bulge near the equator and a flattening near the poles. So maybe this makes a little sense, and the geopotential is a useful concept.

To translate these ideas into a mathematical formulation for a three-dimensional, baroclinic atmosphere, we need to discuss some other things:

- the Coriolis force,
- the geostrophic wind,
- the hydrostatic assumption in equations of motion of the atmosphere, and
- the thermal wind relation.

All these play on the weather map each night in the news as we watch the highs and lows battle it out in the midlatitude Ferrel cell.[13]

5.2 ▪ Spinning Around: The Effects of Rotation

Sometimes on the evening news, they show a globe spinning around, often the wrong way. The sun should rise in the east, which means that, standing at any fixed spot on earth except the poles, we are traveling east. If you think about it, people near the equator are traveling faster, due to rotation, than people at the mid and high latitudes, and we are swinging not around the center of the earth but around the earth's axis of rotation. We are rotating around the line through the poles. The plum line, seemingly at rest, is also rotating around this axis, and that rotation causes the bob to be pulled out of line with the center of the earth. The correction gives an "effective gravity vector" that is different depending on latitude.

The "extra" forces that result from rotation are captured in the equations of motion by an "effective gravity term" and by what are called the Coriolis forces. As these forces increase with increased rotation rate, the dependence of the largest terms in the horizontal equations of motion is on $f = 2\Omega \sin \phi$, where Ω is the earth's rotation rate (in radians per second) and ϕ is the latitude. These extra forces due to rotation cause some strange things to happen, as we will see.

5.3 ▪ Strat-i-fi-ca-tion: Hydro-stat-ic and Geo-stroph-ic

Because air is a compressible fluid with variable density, the higher density gas sinks to the bottom, and the thinner air is left above. This effect, called stratification, allows us to consider the atmosphere composed of layers of air, each of different density. The hydrostatic approximation further captures the stratification of the atmosphere and eliminates all but the biggest terms of the vertical momentum equation. What is left is

$$\rho g = -\frac{\partial p}{\partial z}. \tag{5.1}$$

The pressure gradient balances with the "effective gravity," g, in the hydrostatic approximation, so no vertical velocity is involved in the geostrophic wind.

[13]The Hadley, Ferrel, and Polar cells make up the largest structures of the three-dimensional general circulation of earth's atmosphere. This appears to be secret information, as it is never mentioned on the evening news.

The geopotential gradient and the Coriolis are the dominant terms of the momentum balance in a horizontal atmospheric layer. Equating these two terms, (5.2) gives a definition of the *geostrophic wind*, $\mathbf{v}_g$:

$$f\mathbf{k}\times\mathbf{v}_g = -\frac{1}{\rho}\nabla p, \tag{5.2}$$

where the gradient is the horizontal operator

$$\nabla p = \left(\frac{1}{a\cos\phi}\frac{\partial p}{\partial\lambda}, \frac{1}{a}\frac{\partial p}{\partial\phi}\right). \tag{5.3}$$

The gradient of a scalar function is a vector that points in the direction of greatest increase of the function. On the edge of a high, ∇p points to the center of the high pressure area. Applying the right-hand rule[14] for the cross product, $f\mathbf{k}\times\mathbf{v}_g$, we see that flow is along pressure contours since $\mathbf{k}$ points vertically away from the earth's center. It is clockwise around high pressure where $f = 2\Omega\sin\phi > 0$ in the Northern hemisphere and counterclockwise around low pressure. In the Southern hemisphere, since $f < 0$, the flow directions are reversed. So the geostrophic wind explains what is said every night about highs and lows on the evening news. It also explains why the winds tend to follow the contours of pressure rather than, more intuitively, blowing down a pressure gradient.

Substituting for ρ from the ideal gas law, $p = \rho RT$ (see [46, Eq. (3.86)]),

$$\frac{p}{RT}g = -\frac{\partial p}{\partial z} \implies \frac{g}{T} = -R\frac{\partial \ln p}{\partial z} \implies \int_0^z g\,dz = -R\int_0^z T\frac{\partial \ln p}{\partial z}dz \implies$$

$$\Phi(z) \equiv gz = \int_0^z g\,dz = -R\int_{p_s}^{p(z)} T\,d\ln p. \tag{5.4}$$

The Φ is the *geopotential*, and the hydrostatic equation can be written simply in terms of it as

$$\frac{\partial\Phi}{\partial\ln p} = -RT. \tag{5.5}$$

5.4 ▪ Heat Pumping: The Thermal Wind

Under a geostrophic approximation, the geostrophic wind can be written in terms of the gradient of the geopotential as

$$f\mathbf{k}\times\mathbf{v}_g = -\nabla_p\Phi. \tag{5.6}$$

From the hydrostatic assumption

$$\frac{\partial\Phi}{\partial p} = -\frac{RT}{p}. \tag{5.7}$$

Differentiating the wind equation with respect to p,

$$\frac{\partial}{\partial p}\left(f\mathbf{k}\times\mathbf{v}_g\right) = f\mathbf{k}\times\frac{\partial\mathbf{v}_g}{\partial p} = \nabla_p\frac{\partial\Phi}{\partial p} = -\frac{R}{p}\nabla_p T \tag{5.8}$$

[14] Trying to explain the right-hand rule is embarrassing. Everyone seems to have a slightly different varient, for example, index finger on **a**, middle on **b**, then thumbs up for $\mathbf{a}\times\mathbf{b}$.

leads to the equation for the wind shear in the vertical:

$$\frac{\partial \mathbf{v}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \nabla_p T. \tag{5.9}$$

The way to interpret this is that it gives an indication of the way that the geostrophic (layer) wind varies in the vertical. If the horizontal temperature gradient is large, then the layer wind will increase with height. This equation is used to understand the location of the atmospheric jets between warm and cold air masses. The direction of the jets (west to east in both the Southern and Northern hemispheres) is also given through this equation, with the change of sign in f between the hemispheres being balanced by a change in sign of the thermal gradient. At two different pressure levels p_1 and p_2 in the atmosphere the thermal wind $\mathbf{v}_T$ can be defined as

$$\mathbf{v}_T = \mathbf{v}_g(p_2) - \mathbf{v}_g(p_1) = -\frac{R}{f}\int p_1{}^{p_2}\mathbf{k} \times \nabla T d \ln p = \frac{R}{f}\mathbf{k} \times \nabla \bar{T} \ln\left(\frac{p_1}{p_2}\right). \tag{5.10}$$

This leads to

$$\mathbf{v}_T = \frac{1}{f}\mathbf{k} \times \nabla(\Phi_2 - \Phi_1). \tag{5.11}$$

The thermal wind blows parallel to isotherms, with the warm air mass to the right facing downstream in the Northern hemisphere.

Between the geostrophic and the thermal winds, most of what appears on a weather map can be explained. These brief derivations show that what we see is the result of rotation and stratification of the atmosphere. The temperature distribution that results from the sun and differential heating of the earth acts as a stick that constantly stirs the air and brings us tomorrow's weather.

5.5 ▪ Cartoon Fronts

Another rather curious thing that you will notice on a weather map is the drawing of weather fronts that separate air masses and often define lines of storms. Do these cartoon figures actually correspond to anything in nature? An examination of relations from the geostrophic and hydrostatic equations would yield only a partial mechanism that could cause fronts to develop. Feeding into lows, there seem to develop lines of tightening thermal gradients. The thermal wind relationship predicts that the flow ahead of a (Northern hemisphere) cold front will bring warm, moist air along the line feeding into the low, while behind the front, the flow is in the opposite direction bringing cold, dry air. But this tightening of the thermal gradients and the overrunning of air masses take place on a pretty small scale—a scale smaller than permited in the hydrostatic assumption. A line of clouds often can be seen as a wind picks up from an approaching front. As a tennis player or golfer, one learns to keep an eye on such happenings. But to explain what is really happening, it turns out we need more than just the gradient (∇). The lows suck up surface air and propel it aloft in the atmosphere, while highs bring dry upper air down to the surface. They develop as areas of atmospheric convergence (lows) or divergence (highs). To explain this further, the divergence operator (∇·) would have to be brought into the story, and that is bad news.

Chapter 6

Stochastic Forcing and Predictability

The "butterfly effect" and chaos theory have reverberated through the climate research community since Lorenz discovered it in 1963 [28]. He did it using a very simple model of the thermal atmosphere and noting that small perturbations in the thermal field (like a butterfly flapping its wings) caused the model to follow a very different path with complete separation after about ten days. Since then we have come to understand the "Lorenz attractor" as a kind of phase space climate [22], and many believe, though without proof, that the real climate system must have an attractor that corresponds to what we should call "the climate."

Chaos theory has convinced us not that the problem of predicting beyond ten days is hopeless but that we must try to predict something more general than weather as an instantaneous state of the system. We must try to predict the properties of the system that control its asymptotic behavior. This behavior has two modes, one that arises from the internal variability of the system and another that responds to external forcing. For each of these modes, we would like to understand the bounds on the behavior and something about the probablity of a given behavior occurring. For example, we would like to know how stable the climate system is to stochastic forcing, such as the butterfly flapping its wings. When we look outside, it is apparent that weather is continually perturbing the state of the system with differential heating of the surface triggering baroclinic instabilities and Rossby waves. But it always remains within bounds or settles back to reliable statistics of weather. In this sense, it is pretty convincing that the state of the climate system is near a stable equilibirium and the attractor is reasonably bounded.

6.1 ▪ Uncertainty in Climate Projections

The IPCC AR4 documents refer to uncertainty in climate projections as arising from three sources:

- Parameter uncertainty: the errors associated with data collection and measurement instruments.
- Structural uncertainty: the known and unknown deficiencies in the formulation and solution of model equations.
- Chaotic uncertainty: the inherent limitations of the system to forecasting.

This last uncertainty is what we will refer to as the *predictability of the system*. If the model is represented by a dynamical system of the form

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, \alpha), \tag{6.1}$$

then we consider the growth of error between two solutions starting from slightly different initial conditions at time $t = 0$. Under ideal—perfect model—situations the error could be between the observations and the simulation. But this might be better termed the *skill of the model*, which due to predictability would eventually degrade. The α in this equation refers to controlling parameters of the system as we might be interested in the predictability as a function of certain parameters.

6.2 ▪ Linear Predictability

With $\mathbf{e} = \mathbf{u} - \bar{\mathbf{u}}$ a linear predictability theory examines the growth of

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e}, \tag{6.2}$$

where $||\mathbf{e}(0)|| \leq \epsilon$ and $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\bar{\mathbf{u}})$. If the error grows large over some time interval in response to a small perturbation in the initial condition, then the system is not predictable. But if it remains suitably bounded, then we will say it is predictable with respect to initial conditions. The suitably bounded part is better represented by the logistic equation that Lorenz (1989) used to describe the saturation of average forecast error. With $\mathbf{e}_0$ as the initial error, the error grows according to the equation

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e}(1 - \mathbf{e}). \tag{6.3}$$

The error eventually saturates at 100% or $\mathbf{e} = 1.0$. Lorenz found that for weather prediction $\mathbf{A} = 0.35/day$ gives a doubling time of the error at about two days. Full weather models are not predictable beyond about ten days. But climate is something different from weather, and the $\mathbf{u}$ can be thought of as defining a trajectory in a different phase space than the one used for weather. For example, the phase space might involve seasonally averaged or regionally averaged quantities. We can also project onto the major modes of variability of the climate system that track ENSO, PDO, NAO, etc. The definition of the appropriate phase space is important so that we can be clear about what we are trying to predict. If we try to predict decadal climate, for example, we should choose quantities that vary over that time scale.

To distinguish between the weather noise and longer time scale predictability associated with the boundary forcing of the atmosphere by the sea surface temperatures, the term "potential predictability" or "predictability of the second kind" has been used. Kalnay defines this as the difference between the total variance of the anomalies averaged over a month or a season minus the variance that can be attributed to weather noise. (This is apparently due to Madden (1989).) These definitions are framed by the following theorem.

Fundamental Theorem of Predictability

- Unstable systems have a finite limit of predictability.
- Stable systems are infinitely predictable.

What makes the weather system unstable is the presence of baroclinic instabilities. The intermediate range is unstable because of a variety of instabilities in the ocean and atmospheric circulation patterns. For example, the instability in the Walker circulation in the Pacific gives both the ENSO and the PDO. Longer term instabilities are associated with changes in the Meridional Overturning Circulation (MOC) and changes in land cover, especially snow cover because of albedo feedbacks.

A catalogue of climate system equilibria might be charted with their horizons of predictability and their relative effects on global (or regional) temperature and precipitation. Depending on how near our parameters (such as orbital parameters, or CO_2) place us to a stable or unstable equilibrium, the horizon of predictability will change. The notion of potential predictability can be generalized to specify which time scale and which unstable equilibria are being stepped over. Lorentzian predictability negotiates geostrophic

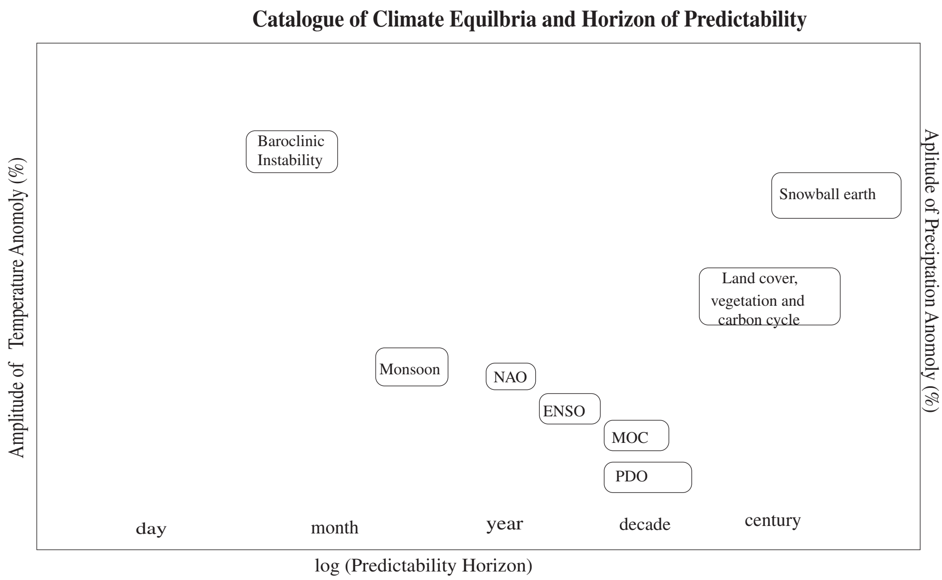

Figure 6.1. *This graph is only qualitative and conjectural for the purpose of illustration. Each type of equilibrium is associated with an instability or adjustment process with a time scale and an amplitude of response. The strength and stability of the equilibrium determine its influence and interaction with a given phase space orbit or trajectory.*

adjustment and the baroclinic instability to reach the predictability horizon of weather systems. Decadal prediction must negotiate the MOC and PDO, treating the higher frequency instabilities statistically as a stochastic forcing.

6.3 ▪ Initial Conditions and Statistical Predictability

The notion of the statistical predictability of a system has been explored by Kleeman [41]. For a system sensitive to initial conditions one is interested in a probability distribution function (PDF) of initial conditions and the PDF of the evolved solution compared to a known (prior) climatological PDF. A measure of the useful information content in the evolved PDF, p, relative to q is given by the form

$$R(p,q) = \int p \ln\left(\frac{p}{q}\right) dV. \tag{6.4}$$

This is known as a relative entropy measure. If the prediction is close to the asymptotic state, then R will be near zero, and the information content of the prediction is small. The measure is based on information theory ideas and is proposed as a replacement of the error measuring the difference between two trajectories sensitive to initial conditions. For decadal predictability, the PDFs of interest would be for the MOC and PDO. A horizon of predictability giving decadal skill would consistently track MOC and PDO oscillations under the influence of a stochastic forcing of ENSO, NAO, and weather. The information content of the generated PDFs would be measured against the climatological PDFs of these same quantitites.

A comment on the influence of initial conditions is in order, as we have been speaking as if stochastic forcing had nothing to do with initial conditions. For the baroclinic instability, a slight change in initial conditions results in a different trajectory, but for monsoonal circulations, this same sensitivity is lost and can be captured as a stochastic atmospheric forcing. Soil moisture and the state of lower frequency oscillations may have a big effect on the monsoonal circulation. In this sense, the initial states of the ocean and land hydrologic cycles are important initial conditions for the decadal prediction problem, while the standard weather conditions or monsoonal heating are not [43]. The climate problem for the century time scale is a boundary (forcing) problem with little or no influence from the initial conditions of weather or short term oscillations.

Fluctuations of the ocean thermohaline circulation occur partly due to anomalous fresh water flux. The phase space map of ocean circulation shows two stable equilibria connected by an unstable branch. The bifurcation diagram of stable ocean equlibrium is given for the first time in [48]. In the absence of the anomalous freshwater flux, they find a single stable equilibrium for the ocean circulation. So ocean predictability is in much better shape than atmospheric predictability. A stochastic forcing of the ocean circulation by freshwater fluxes could hit on a particular spatial pattern that would cause rapid climate change to occur, i.e., a move from one stable equilibrium to another along an unstable path. Alley [2] discusses the contribution of edge ice on Antarctica and Greenland in a stochastic forcing of the Holocene. This raises a number of concerns about abrupt climate change and its relationship to predictability, since abrupt change is something we would like to avoid, and perhaps we could if we could predict it. The equlbria and bifurcations of a simple atmospheric model are considered in [13] and [22]. The predictability of different climate regimes may be the key issue for decadal forecasting, where a regime means a region of the phase space of the system. For low dimensional models, it is possible to map out the phase space. Of course, the different regimes must be clearly identified, a task that seems possible using bifurcation diagrams for idealized systems. The transition between regimes must then be predicted with some skill. For the coupled climate system, of course, we cannot rely on a low dimensional setting. Quantifying skill and uncertainty of longer term predictions will require a systematic investigation into the predictability horizon associated with the equilibria of the climate system.

6.4 ▪ Climate Risk

There is a related concept, that of risk, that should be addressed in the context of climate equilibria and predictability. The risk to human health or risk of exciting a climate shift depends on volatility and nearness of the earth's climate to an unstable point. Like the inverse of the horizon of predictability, the risk is probably low near a stable equilibrium and high near an unstable one. In evaluating the risk of a geoengineering proposal, we would like to be able to predict the consequences, and this is precisely what is impossible near an unstable equilibrium where the horizon of predictability is small. But it is exactly in this situation that the system may need the most active control and the draw of a geoengineering solution is strongest. For example, the CO_2 changes may be pushing us toward an MOC shift, and by increasing the source of atmospheric aerosols in the stratosphere we can induce a cooling and thus a strengthening of the MOC. Such active control has significant risk the nearer we are to the MOC switch. This is the sense of tipping points, the popularized version of bifurcations diagrams. But the mathematical portrait we are painting would suggest that there are many small shifts that might be observed as the parameters change, rather than one big catastrophic point of no return. With each shift, the oscillations reconfigure their interactions to a new orbit around the equilibria points. The risk then comes from the manifestation of smaller shifts in many of the circulation properties. For example, a change in the number of Pacific typhoons may be related to changes in the Atlantic hurricane season. A cumulative risk is needed to sum the contributions of these shifts, taking into account that the horizons of predictability change depending on the state of the climate system.

Chapter 7

Time's Arrow and Methods for Stochastic Differential Equations

In the discussion of the politics of climate change it is hard to tell whether we are moving forward or backward. Since climate modeling is about understanding past climates on the basis of theory and observations as well as projecting future climates, we should talk a little bit about the time dimension. In particular, we will get around to discussing the numerical integration of the equations and the methods that can be used.

The basic formalism for climate modeling and continuous differential dynamical systems in general is

$$\frac{du}{dt} = F(t,u), \tag{7.1}$$

$$u(t_0) = u_0 \text{ for } t \geq t_0. \tag{7.2}$$

The tacit assumption is that we are interested in integrating forward in time. But why not simply substitute $\tau = -t$ to pose a backward in time problem,

$$\frac{du}{d\tau} = -F(\tau,u), \tag{7.3}$$

$$u(t_0) = u_0, \text{ for } \tau \leq t_0? \tag{7.4}$$

The basic existence and uniqueness result for ODEs [11, 4] says that there is a unique solution as long as F is bounded and Lipshitz continuous around the time t_0. F is Lipshitz means that there is a constant L such that

$$|F(t,x) - F(t,y)| \leq L|x-y| \text{ for all } (t,x(t)) \text{ and } (t,y(t)) \in R. \tag{7.5}$$

Since the region R contains the (t_0,u_0), there is a unique solution to the problem in R going both forward and backward. So the formal equation seems to have no preferred direction for time. This could be very useful for climate modeling in order to understand paleoclimates or even verify that we have the right equations to model the present climate. But in climate dynamics we have trajectories that converge to an equilibrium point (in finite time). And there are likely multiple paths to possibly multiple equilibria. So backward in time has a problem with multiple solutions and uniqueness. We can conclude that the F of the climate system is not Lipshitz continuous. This further implies that we cannot guarantee unique solutions forward in time except based on further considerations.

The physicist Steven Hawking expresses the preferred direction of time as the "arrow of time" [21] and attributes its reality to three factors: cosmological expansion, the fact that most of us have trouble remembering things in the future, and the second law of thermodynamics. The second law states that in a closed system, disorder increases with time; that is, entropy is a monotonically increasing function in only one direction. In equation form this is expressed as

$$\frac{dS}{dt} \geq 0. \tag{7.6}$$

The thermodynamic equation of the atmosphere is, in fact,

$$\frac{dS}{dt} = \frac{Q}{T}, \tag{7.7}$$

where Q is the adiabatic heating and T is temperature (in Kelvin so that there will be no division by zero). The relationship

$$dS = \frac{C_p}{T} dT - \frac{R}{p} dp \tag{7.8}$$

brings the standard atmospheric variables of temperature and pressure into the equation. So the thing that keeps us moving forward in time is the differential heating of the earth's surface represented by $Q \geq 0$.

In the atmosphere much of the flow we observe is, in fact, reversible and adiabatic in the sense that $\frac{dS}{dt} = 0$ and nothing like an equilibrium is evident. Weather never seems to settle down much. It would be nice if the numerics we used to integrate the system in (7.2) mimicked these properties, i.e., exactly reversible for adiabatic flows. We would also like to be sensitive to the singularities/equilibria of $F(t,u)$ so that we don't step over (or into) important dynamical processes. If we fail to use numerical methods that respect the underlying dynamics, we may be confusing time's arrow, and we will not know which direction we are going.

There are many ways to think about numerical integration of (7.2), and the rest of this lecture will outline a particular view that I believe is advantageous for climate modeling and possibly other applied fields. It has developed over the last couple of decades with the discovery of *symplectic* methods [37] for Hamiltonian systems.[15]

We make some assumptions about the function F and take an operator theory view in the discussion that follows. Let's assume first that the equation is an (autonomous) linear system (or can be approximated at a given time by a linear system) and can be written

$$\frac{du}{dt} = Au. \tag{7.9}$$

Here $u = (u_1, u_2, \ldots, u_k)^T$ is a column vector with k-components and A is a $k \times k$ matrix. Then the exact solution can be written using the matrix exponential,

$$u^{n+1} = e^{A\Delta t} u^n. \tag{7.10}$$

The matrix exponential is defined by a formal Taylor series

$$e^{A\Delta t} = I + \Delta t A + \frac{\Delta t^2}{2!} A^2 + \cdots, \tag{7.11}$$

[15]Hamiltonian systems develop a flow, $X(t) = \Phi_{t,H}(x)$, that maps the spatial domain to itself. If a surface or volume, Σ, is mapped with the same area or volume measure, $m(\Phi_{t,H}(\Sigma)) = m(\Sigma)$, then the transformation is symplectic. A numerical method that preserves areas or volumes is also called symplectic.

so, for example, an implicit fourth order scheme can be derived from the approximation

$$e^{A\Delta t} \approx \left(I - \frac{\Delta t}{2}A + \frac{\Delta t^2}{12}A^2\right)^{-1}\left(I + \frac{\Delta t}{2}A + \frac{\Delta t^2}{12}A^2\right). \tag{7.12}$$

The abstract formulation with an operator A mapping one Banach space to another is necessary to properly pose the problem of atmospheric or ocean dynamics because we must deal with partial differential equations where spatial scales and time scales are coupled and interact. The eigenvalues of the operator A will typically represent all the spatial scale interactions, and we know even for the Laplacian $A = \nabla^2$ the real parts of the eigenvalues are negative ($\lambda_n = -n^2\pi^2$ for $n = 1, 2, 3, \dots$ if we take the one dimensional Laplacian). So the ODE system has infinite stiffness, and the operator A is densely defined but unbounded.[16] In the abstract setting, this implies that for the problems we are interested in A does not satisfy a classical Lipschitz condition, $||Au_1 - Au_2|| \le L||u_1 - u_2||$.

Exercise. For $Au = u_{xx}$ verify that a Lipschitz condition does not hold by setting $u_1 - u_2$ equal to the nth eigenvector.

7.1 ▪ Consistency, Stability, and Convergence

For a linear method we have that stability and consistency are necessary and sufficient for convergence of the method to the true solution. *Convergence* is usually expressed with a global error $E(t_n) \equiv u(t_n) - u^n$ bounded by

$$||E(t)|| \le C\Delta t^p, \tag{7.13}$$

where p is the order of accuracy of the method and C is a constant valid for all fixed $t \le T$ and $\Delta t \le \Delta t^*$. For the solution of partial differential equations, this result is known as the *Lax Equivalence Theorem* [27, p. 107].

Sanz-Serna and Verwer [38] generalize this result using a *contractivity* property of the solutions. In particular, for a discretization A_h and two solutions starting from two different initial conditions u_0 and v_0, we must have a contraction of the solutions with increasing time, i.e.,

$$||u_h(t) - v_h(t)|| \le ||u_h(s) - v_h(s)|| \text{ for all } t > s. \tag{7.14}$$

The spectral condition on eigenvalues having nonnegative real part is necessary (but not sufficient unless A_h is normal) to guarantee this contraction property. A necessary and sufficient condition for contractivity is that $||e^{\Delta t A_h}|| \le 1$ for all $\Delta t \ge 0$.

With the discretization consistent and contractive, it is then also convergent, and it will then be proper to demand that the time discretization preserve these properties. If the update follows the formula

$$u_h^{n+1} = R(\Delta t A_h)u_h^n, \tag{7.15}$$

then

$$||u_h^{n+1} - v_h^{n+1}|| \le ||u_h^n - v_h^n|| \tag{7.16}$$

if and only if $||R(\Delta t A_h)|| \le 1$. So we have the generalization of Lax's result that *consistency plus contractivity implies convergence*.

[16] Stiffness is usually defined as the ratio of the largest to the smallest eigenvalue of the system matrix A.

7.2 ▪ Structure Preserving

We have mentioned the leapfrog method several times and now will examine its relation to the broader class of methods known as *symplectic methods* [36]. These methods are the major advance in numerical solution techniques and theory in the last several decades. The work of J.M. Sanz-Serna and colleagues on approximation for Hamiltonian systems is credited with the breakthrough [37].

The leapfrog method is analyzed in [17] for a scalar oscillatory equation with $A = i\omega$. In this case the three time level method can be written

$$\begin{pmatrix} u^{n+1} \\ q^{n+1} \end{pmatrix} = \begin{bmatrix} 2\mathbf{i}\Delta t\omega & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} u^n \\ q^n \end{pmatrix}. \tag{7.17}$$

The propagation matrix R has eigenvalues λ satisfying $\lambda^2 - 2\mathbf{i}\Delta t\omega\lambda - 1 = 0$ for which $|\lambda_\pm| = 1$ if and only if $|\omega\Delta t| \le 1$. And if $\omega\Delta t = 1$, then

$$R = \begin{bmatrix} 2\mathbf{i} & 1 \\ 1 & 0 \end{bmatrix} \text{ and } R^n = \mathbf{i}^n \begin{bmatrix} n+1 & -\mathbf{i}n \\ -\mathbf{i}n & 1-n \end{bmatrix}. \tag{7.18}$$

Since $||R^n||$ grows linearly with n, the method is only stable for $|\omega\Delta t| < 1$.

Sanz-Serna asks a further question about methods for specialized types of equations. He asks, "Is there such a thing as a Hamiltonian numerical method?" [37]. A Hamiltonian system looks like

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{7.19}$$

$$\frac{dq}{dt} = +\frac{\partial H}{\partial p}, \tag{7.20}$$

where $H(p,q)$ is the Hamiltonian. The discrete iteration looks similar to the leapfrog system

$$\begin{pmatrix} p^{n+1} \\ q^{n+1} \end{pmatrix} = R_{n,H} \begin{pmatrix} p^n \\ q^n \end{pmatrix}. \tag{7.21}$$

$R_{n,H}$ takes the place of a discrete flow Φ for the system. So the right question to ask is "Are there numerical methods for which $R_{n,H}$ is a symplectic transformation for all Hamiltonians and for all step sizes Δt?" A flow $\Phi_{t,H}$ is a symplectic transformation if $\Phi_{t,H}(\Sigma) = m(\Sigma)$, where Σ is a two-dimensional surface in the domain of $\Phi_{t,H}$ and m is a volume (or area) measure. In other words, the mapping preserves volumes or areas. A discrete Hamiltonian method would then be one with similar preservation of structure.

The leapfrog method, also known as the Verlet method [42], for Hamiltonian systems is both time reversible and symplectic.

Are there other discretizations that are symplectic? The answer is "yes," and we will examine some of the simple modifications of standard methods. More generally, Runge–Kutta methods have been adopted for structural integrity.

7.3 ▪ Stochastic Structure

The question of accuracy of the numerical method appears to be moot if priority is given to preserving structure. There is the famous theorem of Godunov (discussed in Chapter 4) that guarantees that we cannot have everything at once when the solution has discontinuities, for example. What about when the solution has a random, stochastic component? Is it worthwhile to track random, oscillatory behavior at very small time and space scales?

In [7], a harmonic oscillator is perturbed by a random forcing, $\xi(t)$. The equation

$$\ddot{x} = f(x) - \eta\dot{x} + \epsilon\xi(t) \tag{7.22}$$

represents damping with the parameter η and a regular forcing through $f(x)$. The random component is assumed to be normalized by $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. If $f(x) = -V'(x)$ is derived from a potential, then the equation can be rewritten as a position of a particle with deterministic forcing f and a random forcing ϵ related to a temperature by a fluctuation-dissipation relation

$$\epsilon^2 = 2\eta\varkappa T. \tag{7.23}$$

The stochastic system is then

$$\frac{dX}{dt} = V(X,t), \tag{7.24}$$

$$\frac{dV}{dt} = -\eta V + f(X) + \epsilon\frac{dW}{dt}, \tag{7.25}$$

where $\langle W(t)W(s)\rangle = min(t,s)$ is a Wiener process. This system has a probability density at t of

$$P(x,v,t) = \frac{d}{dx}\frac{d}{dv}Prob(X < x, V < v) \tag{7.26}$$

and a stationary density defined as the limit $P_\infty(x,v) = \lim_{t\to\infty} P(x,v,t)$. The analytic solution is

$$P_\infty(x,v) = N\exp\left(-\frac{v^2}{2\varkappa T} - \frac{V(x)}{\varkappa T}\right). \tag{7.27}$$

If this is the true (stochastic) solution for the long time behavior of the equations, should a numerical method for (7.25) preserve the stationary density? Using an approach similar to that of Sanz-Serna, Burrage [7] makes the statement "a good numerical method will be one whose discrete time dynamics has a steady-state density as close as possible to the continuous time system." We describe the leapfrog and the implicit midpoint methods as applied to the system and their resulting stochastic properties for this system.

The leapfrog method for the system updates successively:

$$\hat{X} = X^n + \frac{1}{2}V^n\Delta t, \tag{7.28}$$

$$V^{n+1} = -\eta V^n\Delta t + f(\hat{X})\Delta t + \epsilon\Delta W, \tag{7.29}$$

$$X^{n+1} = \hat{X} + \frac{1}{2}V^{n+1}\Delta t. \tag{7.30}$$

This preserves the exact stationary variance of position ($\sigma_X^2 = \lim\langle X_n^2\rangle$), but the velocity variance ($\sigma_V = \lim\langle V_n^2\rangle$) is an increasing function of the dissipation parameter η.

A small modification due to Mannella [29] corrects this problem giving a method where variances are independent of η. The modification is given by

$$\hat{X} = X^n + \frac{1}{2}V^n\Delta t, \tag{7.31}$$

$$V^{n+1} = c_1(c_2V^n + f(\hat{X})\Delta t + \epsilon\Delta W), \tag{7.32}$$

$$X^{n+1} = \hat{X} + \frac{1}{2}V^{n+1}\Delta t, \tag{7.33}$$

where

$$c_1 = 1 - \frac{1}{2}\eta\Delta t \text{ and } c_2 = \left(1 + \frac{1}{2}\eta\Delta t\right)^{-1}. \tag{7.34}$$

The implicit midpoint rule gives exact stationary variances for all values of η. These are remarkable results, and it is also shown in [7, p. 13] that the implicit midpoint method is the only Runge–Kutta method that preserves exact stationary density for linear equations and all values of damping. It is described by

$$\hat{X} = X^n + \hat{V}\frac{\Delta t}{2}, \tag{7.35}$$

$$\hat{V} = V^n - \eta\hat{V}\frac{\Delta t}{2} + \frac{1}{2}f(\hat{X})\Delta t + \epsilon\Delta W, \tag{7.36}$$

$$X^{n+1} = X^n + \hat{V}\Delta t, \tag{7.37}$$

$$V^{n+1} = V^n - \eta\hat{V}\Delta t + f(\hat{X})\Delta t + \epsilon\Delta W. \tag{7.38}$$

This can be recast in a more efficient iteration for the intermediate position and an explicit update for the remaining equations. The implicit position update is

$$\hat{X} = X^n + \left(1 + \frac{1}{2}\eta\Delta t\right)^{-1}\frac{\Delta t}{2}\left(V^n + \frac{\Delta t}{2}f(\hat{X}) + \frac{1}{2}\epsilon\Delta W\right). \tag{7.39}$$

The remaining update is

$$\hat{V} = V^n + \frac{1}{2}X^n\Delta t, \tag{7.40}$$

$$X^{n+1} = X^n + \hat{V}\Delta t, \tag{7.41}$$

$$V^{n+1} = V^n - \eta\hat{V}\Delta t + f(\hat{X})\Delta t + \epsilon\Delta W. \tag{7.42}$$

The transient time accuracy of these methods can also be discussed in terms of the variance. For the leapfrog method and midpoint rule the error in σ_x^2 is proportional to Δt^2. So these methods not only preserve the stochastic structure in the long time but are second order during the transient. In atmospheric integrations the leapfrog method is often supplemented with the use of the Asselin time filter. This time averaging acts as a diffusion term, and it is worth noting that accuracy is not affected in the stochastic setting by this diffusion. Some of these properties probably account for the popularity of the leapfrog method in atmospheric modeling even though we may not have known the theoretical underpinnings until recently. They also illustrate that the concept of accuracy can be extended to stochastic equations in a meaningful way.

Exercise. Write a MATLAB code implementing these three methods and compare accuracy with and without noise for different values of damping.

Chapter 8

Mitigation: How Much Climate Change Can Be Avoided?

This lecture is based primarily on Warren Washington's recent GRL paper [45] and the PNAS paper of Auroop Ganguly [20].

Dr. Washington's climate change study gets to the bottom line of a greenhouse warmed future but on a hopeful note; it is the most optimistic future as far as climate change goes that can be imagined. Of course one overlooks much in the way of political practicality or feasibility or even common sense when you put on rose-colored glasses. But Dr. Washington is nothing if not an optimist, and he argues that it is possible to "keep the probability of exceeding a warming of $2°C$ at a third or less." With this the Arctic and permafrost areas would be largely preserved, and, compared to a business as usual scenario, heat wave intensity and sea level rise would be roughly half of what is expected.

Let me stop here and explain some terminology. To mitigate is to avoid or alleviate a more serious condition, and avoiding climate change is largely a matter of taming the increase in concentration of atmospheric carbon dioxide. There are several ways to do this, but more on that later. Mitigation is usually juxtaposed with adaptation; one seeks a cure, while the other is a strategy for living with the pain. Mitigation is an important concept for climate change, as it refers to what might be done to stop climate change for good; problem solved.

It is conjectured that the earth was in a stable condition before the industrial revolution with atmospheric CO_2 at roughly 280ppm. The present level is about 400ppm, and Dr. Washington's optimistic future puts a cap on the eventual concentration at 450ppm by 2100. The clock on the $2°C$ rise started ticking around 1900, and we have already used up $1.4°C$ with $0.6°C$ left. A new stable equilibrium would not be reached for several hundred years even though the CO_2 concentration would be stable by 2100. This is because the oceans and terrestrial ecosystems adjust to the changing conditions very slowly. It will likely take another couple of centuries for the ecosystems to adjust to the new climate zones, and ocean turnover for the deep ocean to adjust to increased temperature in the surface mixed layer is probably five to ten centuries.

The main point of Dr. Washington's paper, however, is not the consequence of climate change; this has been covered rather exhaustively in the 2007 Intergovernmental Panel on Climate Change (IPCC) AR4 report [31, p. 104], and the story has not changed much since the 1990s. Rather, the point is to quantify the consequence of inaction—of not attempting to mitigate climate change. The comparison is not with the present or even preindustrial climate so much as it is with a nonmitigating option. In the nonmitigating option the CO_2 concentration rises to about 750ppm by 2100 and presumably would continue to increase after that. So the mitigated case compares the equilibrium as partially achieved by 2100 with the nonequilibrium conditions of a continued increase in atmospheric CO_2.

8.1 ▪ What Happens If We Lower Emissions?

The study finds that the global average surface temperature over ocean and land at 2100 will be about 1.6°C higher between the two options with about an 8cm difference in sea level rise. So the amount of change we experienced in the last century is comparable to the amount of change we can avoid by pursuing mitigation strategies. This study is probably on the conservative side of modeling studies for a couple of reasons. First, the model used is the Community Climate System Model (CCSM3), which has a relatively low climate sensitivity (2.7°C for CO_2 doubling measured in the traditional 1980s way). Second, the nonmitigation scenario concentrations are already lower than has been observed after the rapid growth.

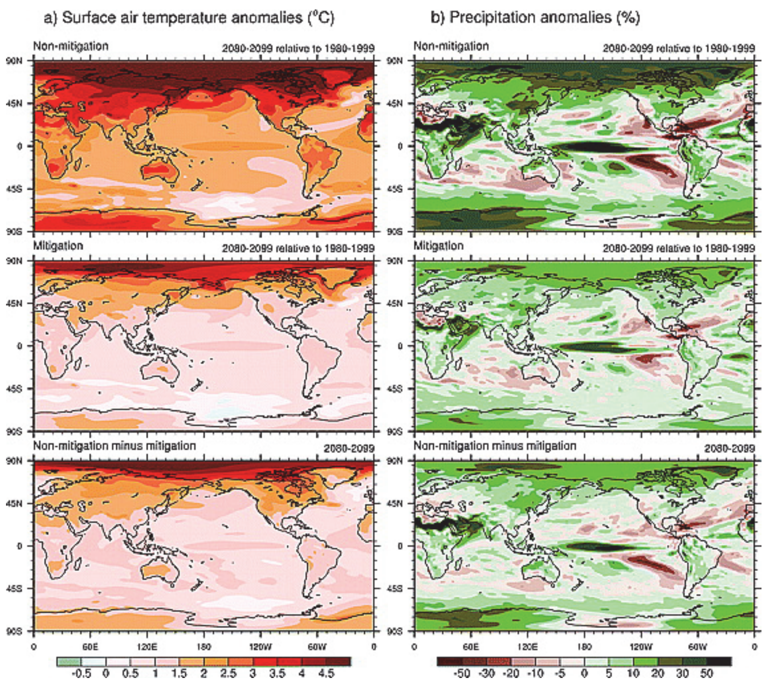

Figure 8.1. *(a) Surface temperature and (b) precipitation changes for the end of the 21st century (ensemble average for years 2080–2099) minus a reference period at the end of the 20th century (ensemble average for years 1980–1999) from 20th-century historical simulations with natural and anthropogenic forcings. (Bottom) Nonmitigation minus mitigation to show warming avoided (Figure 2a) and precipitation change avoided (Figure 2b) [45].*

The hydrological cycle increases in intensity in a greenhouse warmed world due to the ability of warm air to hold more moisture. The consequence is a change in precipitation patterns and amounts. First, the tropics see an intensification of rainfall, especially over the pacific ITCZ and interestingly over Africa and the Arabian peninsula. The Mediterranean, Caribbean, and Central American regions also see a significant drying (20%–50%). The Western U.S. and Southern Australia also see some decrease in rainfall. But large portions of the high latitude areas see significantly more precipitation ranging from 10% in the mitigation scenario to 50% in the nonmitigation case.

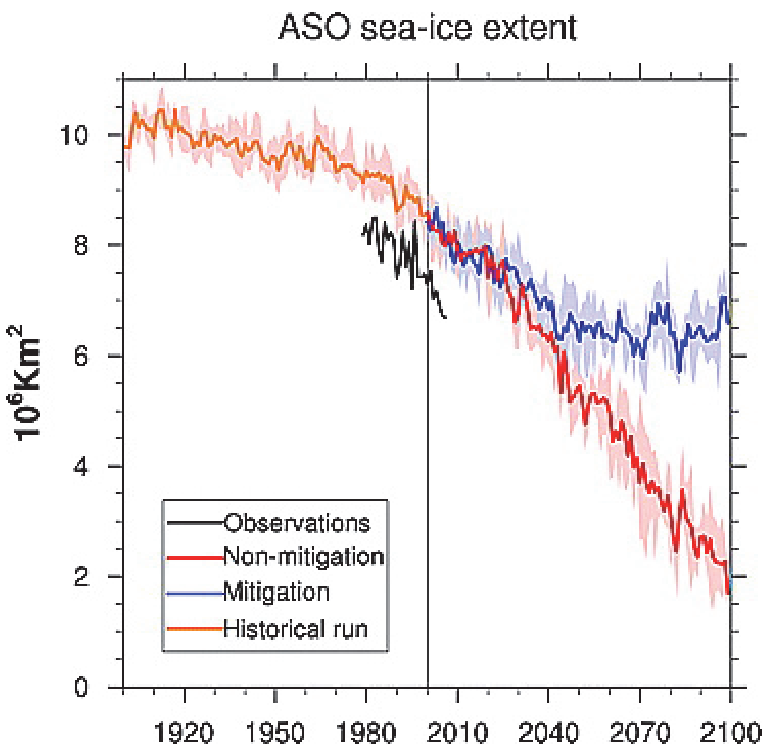

Figure 8.2. *Time series of August, September, and October (ASO) season average Arctic sea ice extent. The dark solid lines are the ensemble means, and the shaded areas show the range of ensemble members. The observed sea ice is shown in black, the historical simulation in orange, the nonmitigation in red, and the mitigation in blue. The model shows a systematic positive bias of about 5% compared to the observations, but the current trends are similar. In the mitigation simulation the sea ice extent stabilizes in the second half of the 21st century, while the nonmitigation simulation decreases markedly in the latter part of the 21st century. Values for just the month of September (not shown) provide a smaller sea ice extent than a three-month average [45].*

Finally, the hot-button area of the Arctic needs to be considered. In the northern hemisphere summer (August, September, and October) the ice is at its minimum extent. Of course, it regrows in the winter. Since the start of the satellite record in 1979 there has been a consistent way to see the ice extent, and observations show a trend of decline. Model results, based on the CCSM3 equilibrium in the preindustrial period when we do not have observations, would suggest that this decline has been going on for some time at a rate comparable to the observations. This is a different rate than would be expected from the slow climate changes associated with orbital changes and the last ice age. The difference between the mitigation and nonmitigation cases shows a very different response in the Arctic and hence a much different response of the sea-ice extent. With mitigation, the corner is turned by about 2040, while without mitigation the summer ice drops precipitously toward zero by the end of the century. This may be one of the most clearly observable signals of global warming and of how well model predictions are capturing the energy state of the earth system. What is interesting about the state of the Arctic ice is that it provides a good indicator of the state of the entire system, just as an ice cube in a glass of water is a good indication that the temperature of your drink is what you expect. But also, a large shift in the energy absorption part of the Earth's radiation budget will occur if the Arctic melts. The change in color from ice to open water changes the reflectance, the albedo, of the surface. The darker color will absorb more incoming solar, heating the ocean faster. These ideas about the radiation budget are the mathematical and physical principles on which the CCSM3 and other climate models are built.

The other positive feedback discussed in the paper deals with the 2.7 million square kilometers difference in permafrost area between the mitigated and nonmitigated scenarios. There is a concern that melting permafrost will allow release of methane trapped in the soil. And methane in the atmosphere is even more potent than CO_2 as a greenhouse gas. So global warming would accelerate according to our physical understanding if the permafrost melts.

8.2 ▪ What Happens If We Don't?

A more severe picture emerges if you consider the path we are currently tracking. That path is even worse than the "nonmitigation" scenario. In fact, the figures we will discuss now are the most pessimistic future yet to be studied. These are based on the A1FI scenario that is analyzed in Auroop Ganguly's recent paper in PNAS cited above. Under this scenario, the future emissions continue apace with a developing world using fossil fuels. Population assumptions for this scenario have the global count leveling off and starting to decrease by 2080. Because this is most severe, an attempt has been made to provide "error bars" and "uncertainties" along with the projections. I hope you won't be misled into thinking these communicate errors in the sense of what is right and what is wrong. Uncertainty is a statistical concept with a precise technical meaning, but that is for another day.

Figure 8.3 shows the global temperature increase studied in the Washington paper. Figure 8.4 has several scenarios, including the red curve that corresponds to A1FI. The global annual average temperature increase exceeds $10°C$ by 2100. The atmospheric CO_2 concentration is 980ppm and continues to rise at the end of the century.

Because the historical variations and amounts do not exactly match the model results and there is an inherent variability in the model, there is an attempt to bound the possible futures in a statistical sense. The panels of Figure 8.5 show the most likely, worst, and best cases within those bounds. The warming in high latitudes for the worst case exceeds $20°C$ in some locations. But in the best case, there is even some cooling over the oceans. So what should we make of this kind of information?

You might be tempted, like the local news, to summarize "Oak Ridge scientists say global warming patterns uncertain" (WBIR TV 9/14/2009). But I think it would be best to view this as new information for the climate change discussion—information that constructs upper and lower bounds instead of just the most likely. This allows you to evaluate the implications of mitigation and nonmitigation in terms of the risk you are willing to take. It is an acknowledgement that most people, and civil infrastructure planners in particular, already know how to evaluate and deal with risk. The bounds allow one to say, "Here is the most likely, but you may get lucky and nothing much will happen." On the other hand, you might be making a really big mistake. This is new research and a new way of presenting climate change information, so not all of the kinks are worked out. But it

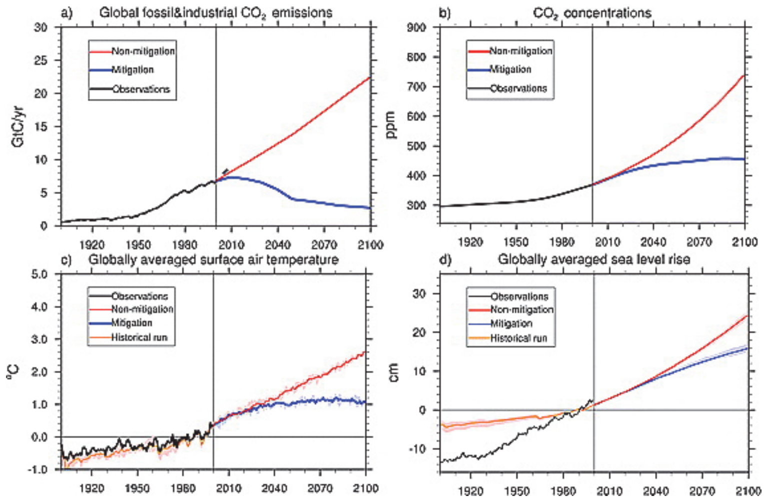

Figure 8.3. *(a) Mitigation (blue) and reference nonmitigation (red) time series of global CO_2 emissions, (b) CO_2 concentrations, (c) globally averaged surface air temperature anomalies calculated from a 1980–1999 reference period, (d) globally averaged sea level rise anomaly (thermal expansion only) calculated from a 1980–1999 reference period. The estimated CO_2 emission and observed concentration data are in black. The mean historical (1900 to 2000) simulation is shown in orange. Note the small dots in Figure 1(a) above the red curve after the year 2000 show the 2005–2007 actual CO_2 emissions [Raupach et al., 2007]. The nonmitigation scenario data is less than actual emissions. The range of individual ensemble members is shown in light shading for globally averaged surface temperature and sea level rise. The observed total sea level rise is shown in black in Figure 1(d) [Church et al., 2001][45].*

is following up on work that has started to characterize the extremes we might encounter in climate change and communicate the uncertainties more effectively.

Since the upper bound and lower bound are typically not shown in discussions of climate change, the reaction of the reporters is understandable. A reaction of fright or panic may also be understandable when considering the upper bound, worst case, result. The headline could have been "Scientists try to scare us with global warming results." Both reactions are counterproductive as far as I am concerned because they fail to grasp the need for actions that mitigate climate change; one is a noncommittal wait and see attitude, and the other is a helpless, apocalyptic stance.

I'm forced to retrace my steps, returning to a perspective that we can all agree on and that gets at the issue with more certainty and less sensationalism.

8.3 ▪ Carbon Balance

The fundamental notion underlying the mitigation discussion is conservation of mass. The fundamental physical laws are posed as conservation properties of mass, momentum, and energy, and climate models solve the equations numerically that express these laws. In mathematics there are several ways of looking at conservation, and one is the notion of an invariant manifold. Think of a manifold as a multidimensional surface, a set of points in a space of dimension larger than three. A set S is invariant with respect to some action $A()$ if $A(S) = S$. Now if we think about the conservation of global carbon, there are only a few

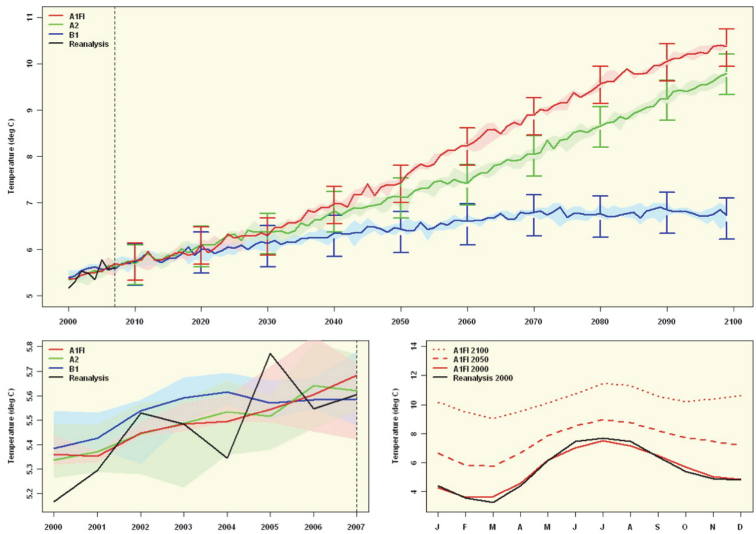

Figure 8.4. *Ganguly et al. (Figure 1). Global average projections of temperatures and uncertainty. The top panel shows globally average temperature (degrees C) projections from CCSM 3.0, based on A1FI, A2, and B1, along with error bars. The bias and standard deviations are calculated for each projection by comparing NCEP reanalysis data with model outputs in 2000–2007, which forms the basis in the generation of the error bars for 2010 to 2100; note that the error bars are based solely on this bias and variance but do not take into account the effect of projection lead times. The shaded areas indicate uncertainties caused by five initial-condition ensembles. The bottom left panel zooms in on 2000–2007. The bottom right panel shows monthly global average temperatures [20].*

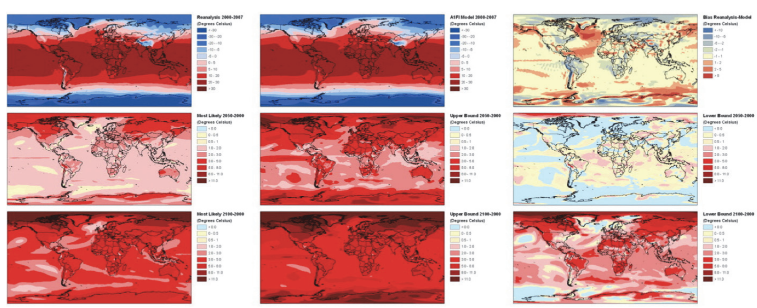

Figure 8.5. *Ganguly et al. (Figure 2). Grid-based temperature projections with confidence bounds for A1FI. The top panel shows reanalysis and model-simulated annual average temperature (degrees C) along with the bias for 2000–2007. The bottom panels show 2050 and 2100 temperature projections from A1FI-forced CCSM 3.0 after bias correction (most likely maps, left) as well as upper (center) and lower (right) bounds. The numbers can be used to support local to regional scale analysis of climate change and extreme hydrometeorological stresses or impacts [20].*

places carbon can be hiding. It can be in the atmosphere as CO_2, in the ocean as dissolved organic or inorganic CO_2, in organic or inorganic forms on land, in leaves, branches, and roots, in liter pools, etc. And of course it can be in our geology in rocks or as oil, coal, and natural gas. There is only so much carbon, and it is not easily created or destroyed, though it can change form in terms of what it is bonded to chemically. It can also move around between its various pools.

Let our space for action be the transfer of carbon between the carbon pools, so that $s \in S$ with $s = (c_1, c_2, \ldots, c_P)$. The c_i is the amount of carbon in pool i, and we count P distinct pools. So the state of the earth's carbon at any given instant is a point s on the manifold S. That carbon is conserved means that $\sum_{i=1}^{P} c_i = C$, the total amount of carbon, for each $s \in S$. The manifold is the surface that the system is stuck on. No action A can move a point off the manifold; conservation is a hard constraint—a certain bound for our discussion.

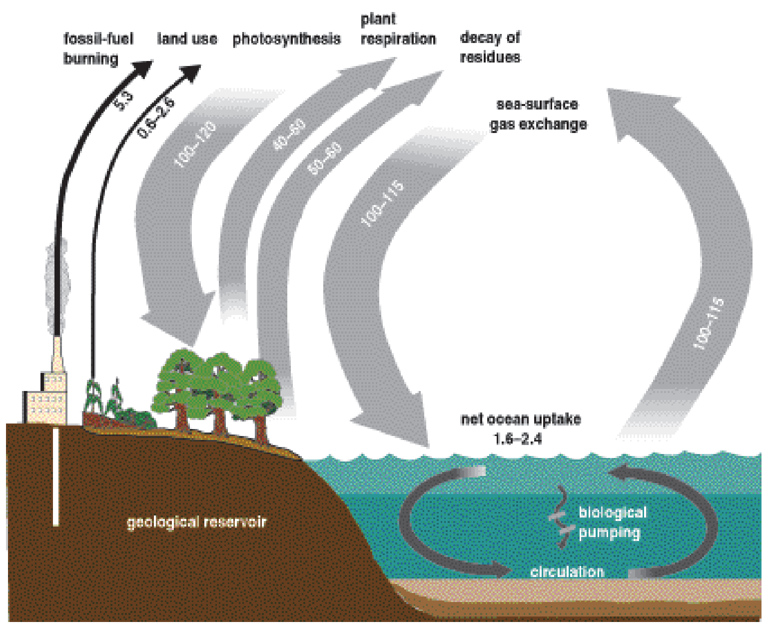

Figure 8.6. *The earth's carbon cycle. Carbon cycles through pools or reservoirs of carbon on land, in the ocean, and in sedimentary rock formations over daily, seasonal, annual, millennial, and geological time scales [24].*

The mitigation actions we can think of to cap the amount of carbon in the atmosphere below 450ppm will require changes or stabilization of the other carbon pools. The climate change dilemma is primarily a result of an out-of-balance carbon cycle that needs to be stabilized so that the other components of the climate system can find a new equilibrium. To this end, it is important to ask lots of carbon questions. What is the state of the carbon pools? How much carbon is in each? How much fluctuation is seen in the natural system and on what time scales? What controls the flow between the pools? How is it possible to change the flows between the pools? Are emission controls the only realistic way to accomplish mitigation? Is the only solution for the energy complex to stop using carbon and let the natural pools equilibrate?

One of the best recent reports on this is [24]. Many of the details of the pools and what is known about rates of exchange between them are discussed. Clearly, the emissions from burning of fossil fuels is the focal point of the discussion. But there are other major players, such as the changes in land use and carbon fluxes from soil as a result of agricultural and forest management practices. The bottom line is that humans have become the carbon stewards for the planet, whether we like it or not. Carbon management is the vehicle we are driving into the future, and pretty soon we need to learn how to steer.

How locally can this be taken? The report I've cited gives details for North America. What about Tennessee? There is not a similar document, but here are some interesting facts. According to the 2008 coal mining statistics [Report No: DOE/EIA 0584 (2008)], all the mines in Tennessee produced 2,333 million (short) tons of coal. Interestingly, Campbell County, Wyoming, produced more than any other state with 415,924, nearly half the U.S. total of 1,171,809. The price charged for electric power producers was about $2.25 per million btu's. How much carbon is in a ton of coal? About 70%, and then when that carbon gets combined with oxygen to form CO_2, the proportionality is roughly that 1 unit of coal turns into 2.93 units of CO_2. The equivalence between energy production and emissions is 1–1.5 kg CO_2/kWh for coal-electric power. Someone once said to me that as long as there is a way to burn something for heat, we will do it.

In an announcement dated October 21, 2009, the following was described: *DOE Partnership Completes Successful CO_2 Injection Test in the Mount Simon Sandstone*. "In the controlled test, members of the MRCSP research team injected liquefied CO_2 at Duke Energy's East Bend Generating Station, located along the Ohio River near the town of Rabbit Hash in Boone County, Kentucky. The CO_2 was injected into the lowest 100 feet of the Mount Simon Sandstone, which is present at the East Bend site at approximately 3,230 to 3,530 feet below ground. The formation has properties that are considered conducive to CO_2 storage, such as the appropriate depth, thickness, porosity, and permeability; in addition the formation is overlain by layers of low-permeability rock that should keep the CO_2 safely and permanently confined."

"Before drilling the test well, the partnership conducted a seismic survey at the site and obtained permits for the injection test from the U.S. Environmental Protection Agency (EPA) and the Kentucky Division of Oil and Gas. The research team then injected clean brine, as required in the permit issued by the EPA, to determine formation properties such as the maximum safe injection pressure. Following brine injection, a total of approximately 1,000 metric tons of CO_2 were injected in two 500-metric-ton steps, concluding on September 25th. The injection rate, pressure, temperature, and quantity of CO_2 in the formation were measured throughout the test to confirm that the injection proceeded as planned."

The facts should not be denied; nor should hopes be placed in unrealistic solutions. What do you make of our carbon management options for mitigating climate change?

Chapter 9

Adaptation to Climate Change

The implications of climate change are not entirely clear. But if we could understand how we are likely to be most affected, perhaps we could do something to avoid the damage or take advantage of it, if you want to look for opportunity amidst chaos. Several years ago I started to get questions about how soon the Arctic would melt because the opening of the Northwest Passage could be a boon to commercial shipping. (So far only eco-tourism has taken advantage: http://www.victory-cruises.com/arctic_nwpassage.html.) The term adaptation is being used in the climate discussion with a sense of inevitability—that climate change is unavoidable and we will need to get used to it. So for this lecture we will try to understand what we might need to adapt to. Climate results have to date been primarily focused on the mitigation question of what we must do to stop or slow climate change. The adaptation question is harder, and this will only be a small step in the direction of what is needed for the science to support decision makers, planners, and engineers. The objective here is to prepare, not to scare.

9.1 ▪ Extreme Events: Heat Waves

Instead of average global quantities, we will primarily be interested in extremes. Extremes of weather are a part of climate if you define climate as the statistics of weather. Characterizing the extremes and the changes in the extremes is only now becoming possible in climate science because of the increased detail of climate models. For example, heat waves are one of the important extremes to consider. The intensity of heat waves and their duration are shown in Figure 9.1 to be changing. Deep red patches for the last decade of the century are noticeable in the Midwest U.S. and Brazil. Parts of Africa and the Middle East also show increases of over 6C in average intensity. Australia shows similar intensity, though less increase over the present day. You might use this graph to make statements about how locations compare for an evening time game of tennis. New England in 2100 would resemble Texas or Oklahoma. Texas and Oklahoma in 2100 would resemble India today. How often heat waves occur, their expected frequency, is shown in Figure 9.2. Engineers want to know the return period for an extreme event, and the graph indicates a dramatic decrease in the number of years between heat waves under the A1FI climate change scenario. It is these kinds of statistics[17] that could prompt changes in building

[17] The statistical methods for analyzing extreme events focus on the tails of distributions and how these might change shape as a result of climate change. For a discussion of return periods and other characterizations of extremes, see [1].

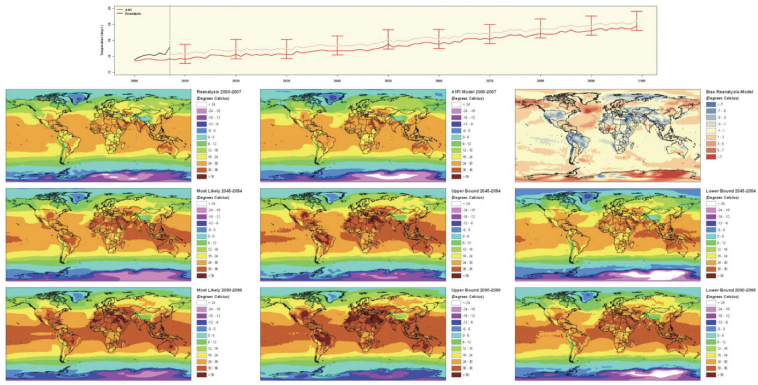

Figure 9.1. *Intensity of heat waves from A1FI. A heat wave is defined as the mean annual consecutive 3-day warmest nighttime minima event. The top two panels show intensity, graphically and mapped, from reanalysis data and model outputs for 2000–2007 along with the bias. The bottom panels show 2050 and 2100 heat wave projections from A1FI-forced CCSM 3.0 after bias correction (most likely maps, left) as well as upper (center) and lower (right) bounds. The numbers can be used to support local to regional scale analysis of climate change and extreme hydrometeorological stresses or impacts [20].*

codes and construction practices as well as measures to protect vulnerable populations in the next few decades.

9.2 ▪ Avoidance Strategies

So what kinds of things can be done to avoid the impacts associated with these increasingly likely extreme events? The basic response, common with animals trying to avoid impacts, is to run and head for higher ground. It is said that the elephants in Thailand and Sri Lanka broke their chains and walked away from the ocean several minutes before the Banda Aceh tsunami hit in 2004. Similarly, the highest regions of the Appalachians and the Rockies (for example, in the Great Smokey Mountains and Glacier National Parks) are a refuge for many species that populate at lower altitudes in times of less stress. These areas are usually the headwaters of rivers feeding large regions and provide a natural corridor for wildlife to travel along. Consequently, these important areas of biodiversity provide the seeding for replenishment of wildlife when threats subside. A list of the World Biosphere Preserves[18] shows an extensive international effort to protect some of these remote areas, and this may have some importance for preserving biodiversity in the face of climate change.

Our own avoidance response may not be to seek refuge on top of a mountain, though mountain hill stations have always been a favorite spot to get out of the summer heat. Humans also have the ability to adapt using technological means. For example, we can develop air conditioned public spaces for vulnerable people, usually the young, elderly, and/or poor, to escape heat waves. We can and do mist livestock with water, providing evaporative cooling in situations of duress. Road beds can be built higher or with increased drainage capability to avoid problems with more intense rainfall events (see Figure 9.2) or to avoid sea level rise and storm surges. And, of course, we can avoid rebuilding

[18] see http://www.unesco.org/new/en/natural-sciences/environment/ecological-sciences/biosphere-reserves/world-network-wnbr/wnbr.

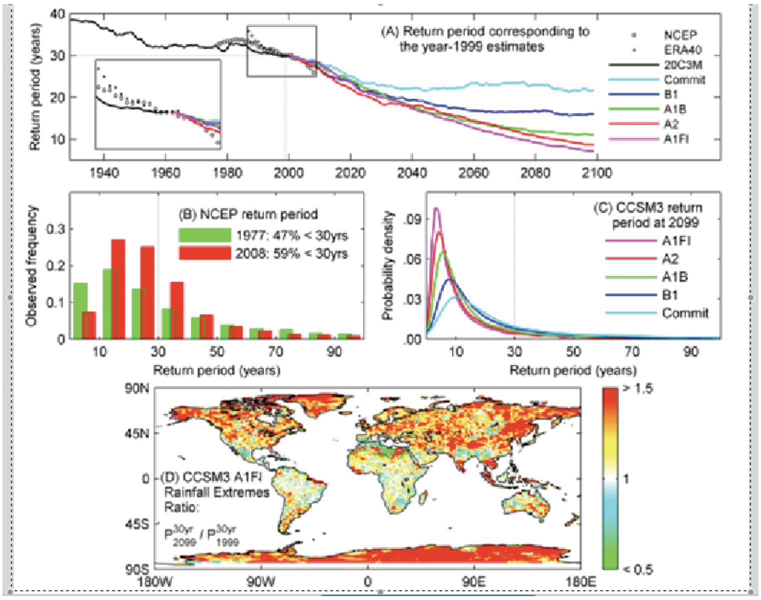

Figure 9.2. *A1FI return periods [20].*

low lying areas that are prone to flooding, as has started to be done along the Mississippi river and its frequently flooding tributaries.

Another strategy for adapting to climate change is to modify our behavior, something we do almost without thinking about it. For example, if the temperature is above 100 ^oF (38 ^oC), the golf course or tennis courts will not be getting much use. We will simply stay indoors more often or change the time of day for being active outside. Cultures around the world already reflect these kind of adaptations to their local climates in the way they conduct business and leisure.

A sense of severity needs to be injected into the discussion of climate change impacts because some of the changes do not seem on par with others. For example, if the measure of severity considers mortal effects, such as death from heat waves, this is not on par with the economic impacts of fewer rounds of golf. So I'd suggest we triage the impacts of climate change and concentrate first on developing adaptation strategies for the worst effects. Health impacts are just starting to be studied and understood related to climate extremes, although the connection of insect-borne disease has a long history with recommended actions. Water shortages to the human population and then to industrial and agricultural systems are perhaps the next most severe impacts. The business community has a vested interest in understanding these and developing appropriate adaptations. Air quality is another impact whose connections with climate change are just starting to be understood, but that has a long history through the Clean Air Act in the U.S. of warning systems and adaptive actions.

How quickly should we respond with adaptive actions to the observed and forecast climate impacts? How fast should we run? This is actually a critical question in the discussion of adaptation because some changes will come faster than others. Suppose we answered this question based only on using observed climate changes.[19] It takes 20 to 30 years of observational data to identify climate trends, and even then, there may be decades in which locally that trend is reversed. So serious damage may occur before there is a definitive documentation from the observed data. On the other hand, many of the adaptation strategies can be implemented fairly quickly. Even building higher road beds could be merged with routine maintenance schedules with only a few years lag time. The shift in frost dates that has modified planting schedules is another slow change that allows a quick adjustment. Doing adaptation on an "as needed" basis may cost more, but it has the advantage of sidestepping the entire climate change issue and being practically motivated from local concerns.

Unfortunately, not all of the adaptation actions can be implemented on short notice. For example, the management of forests and lumber production requires the lead time associated with growth of a tree, often more than the 20 or 30 years of climate averages. For the longer lead time adaptations it would be advantageous to have reliable decadal forecasts of climate change. These, of course, cannot be provided yet by the science. Instead, the best projections are in terms of 20 to 30 year averages based on particular concentration pathways or climate scenarios. The best projections of climate change for the next 50 years indicate that much work needs to start now with the threat being reassessed every decade or so. In this category are flood control, some human health concerns, water resource planning, and biodiversity preservation. Building codes also need modification since the life of a building is 50 to 100 years and retrofitting for more extreme weather is difficult. Current climate simulations should be able to provide reasonable and appropriate bounds on weather and climate parameters used in civil and environmental engineering.

9.3 ▪ Adaptation as a Local Concern

Adaptation decisions must play out in particular regional and political situations. Not only are the climate conditions of each locale different, but the resources and response to priorities is highly dependent on the people and interests involved. Adaptation decisions start with stakeholders and the preceived harm or benefit that results from the local climate change. This local discussion may be somewhat like the emergency preparedness planning that took place as part of the Homeland Security response to the 9/11 terrorist attacks. Even on a local level, people were trained to know what to do when. And this involved not only police and emergency response personnel, but also planners and citizens. An awareness and expertise were built up that could be shared with other localities and supported with resources on the local, state, and national levels.

What would a climate action plan look like on the local level? Since there are now many examples of these[20] as well as advice from the U.S. National Assessment of Climate Change Reports,[21] there is no need to provide much here. Simply put, these reports focus on identifying the local vulnerabilities and concerns and prioritizing a set of actions that would deal with these concerns. Often they are developed in the context of recommendations from a group of stakeholders and climate experts. But as the climate impacts are better understood and quantified, the climate experts play less and less of a role, and engineers will take over as the local providers of expertise and solutions. For this reason, it is important that engineers become familiar with the impacts of climate change and begin to provide revised safety estimates and best practices for dealing with extreme events and

[19] This is the position of the North Carolina legislature, who in 2012 banned the use of model predictions of sea level rise in planning coastal development.

[20] For example, "The Chicago Climate Action Plan"; see http://www.chicagoclimateaction.org/.

[21] see http://www.globalchange.gov/what-we-do/assessment.

a changing climate.

But there are other, nonexpert resources that are also local. The character of a community may track a self-sufficiency curve with a culture of neighbors helping neighbors and a resilience to challenging times. The practices of sustainable development are expressed in the local environment in ways that can complement climate change planning. There may even be religious traditions, the practice of the Jubilee comes to mind, that can help a community deal with changing conditions and the inequities that are sometimes created. Adaptation comes down to practical, everyday stuff like getting your tractor unstuck or providing shelter in times of need. It is about avoiding getting knocked down and building communities to help those who are affected by the changes.

Chapter 10

Energy for the Long Haul

Given the ups and downs of energy supply and demand, we can start this lecture by saying that "Energy" is a very complex subject. I've worked for the Department of Energy for most of my career, have seen many of the best and brightest proven wrong about energy futures, and have seen many great ideas get stuck on a technical glitch or an economic obstacle. Just look at the history of nuclear energy in the USA and its growth as a peaceful and hopeful application of weapons technology from the Manhattan Project. Public opinion (some say public hysteria over nuclear anything) put the industry into a 30 year hiatus from which it may never emerge. But perhaps that is also due to the high costs, complicated and temperamental controls, and regulation that are required to keep a reactor safe? Probably so. Now nuclear power is being touted as a greener and cleaner alternative to coal fired power plants. The French government announced plans to close half of their coal power plants by 2015 [32], citing climate change concerns. For future energy supply there could be fusion energy, the incarnation of the sun in a terrestrial bottle, pollution and CO_2 free. But maybe we should first consider things we know. Compared to these high-tech solutions, bioenergy is decidedly low-tech, safe, and predictable. It may also be ineffectual for the high energy demands of a Western lifestyle. As any deployment of technology has unintended consequences, there is a question of how biofuel generation might be sustained for the long haul, where by long haul I'm thinking about some limit as $t \to \infty$.

10.1 ▪ Models of Competition

Biomass production is in competition with food crop production and other interests such as preservation of forests and biodiversity. If we just restrict ourselves to agriculture and let B be the biomass production and C be the crop production, then the competition for arable land would imply a competitive model such as the famous Volterra–Lotka equations, otherwise known as the predator-prey model,

$$\frac{dB}{dt} = B(\alpha - \beta C) \text{ and } \frac{dC}{dt} = -C(\gamma - \delta B). \tag{10.1}$$

With a dynamical structure like this, what could sustainability mean? As time progresses, the solutions to the equations fall into distinct classes. You might hope for a stable, fixed point solution (B^*, C^*) that does not change in time. Well, that would mean that $\frac{dB}{dt} = \frac{dC}{dt} = 0$. From this condition a solution is $B^* = \frac{\gamma}{\delta}$ and $C^* = \frac{\alpha}{\beta}$. But there is a broader class of "solutions" we can discover by considering the system matrix

$$\mathbf{A} = \begin{pmatrix} \alpha & -\beta B^* \\ \delta C^* & -\gamma \end{pmatrix}. \tag{10.2}$$

The equilibrium solution is in the null space of the system matrix, and for the values B^* and C^*, the system matrix is singular. A test for that would be that the determinant of the 2×2 matrix is zero, i.e., $0 = -\alpha\gamma + \beta\delta B^* C^*$. Solving this shows that a fixed point solution, not equal to zero, will have $B^* C^* = \frac{\alpha\gamma}{\beta\delta}$. Fixed points can be generalized into singular points by considering B and C as complex numbers.

The linear system with $\mathbf{A}$ has solution (see [25, Thm 11.2])

$$\begin{pmatrix} B(t) \\ C(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} B(0) \\ C(0) \end{pmatrix}. \tag{10.3}$$

If A is a diagonalizable matrix, then there exists a matrix X such that

$$X^{-1}\mathbf{A}X = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2). \tag{10.4}$$

Then the matrix exponential can be explicitly computed [25, Thm 9.20] as

$$e^{t\mathbf{A}} = e^{tX\mathbf{A}X^{-1}} = Xe^{t\Lambda}X^{-1} = \sum_{i=1}^{2} e^{t\lambda_i} x_i y_i^H. \tag{10.5}$$

The eigenvalues λ_i are likely complex since A is not likely to be a symmetric matrix. Hence the solution to the differential equation may have terms like $e^{t\lambda} = e^{t(a+\mathbf{i}b)} = e^{at}(\cos tb + \mathbf{i}\sin tb)$. The solution we are interested in would be the real part of this, and it would involve a periodic or oscillating interplay between B and C, a limit cycle. So here are two ideas of what we might mean by "the long haul" and sustainability. Sustainable doesn't necessarily mean static but might involve a stable time variation that stays within some bounds.

10.2 ▪ Biofuel

One of the things you see traveling in India is the use of biofuel for making construction bricks and for cooking and heating. Bricks are fired in a kiln stoked with sugarcane stover (leaves, branches and bark discarded after stripping the cane). Dung cakes are useful for cooking and heating. This has been a source of energy for centuries and may be considered an example of a sustainable practice. Cow manure is mixed with straw and formed into cakes which are dried and stored in small structures for later use in cooking. The cow is a vital link in the processing of biomass into usable human biofuel. In these examples, the biomass crops are not grown separately from the food crops but are part of an agricultural waste stream, which is a very efficient arrangement. In the U.S., where corn production for ethanol currently competes with corn production for food, the situation is less efficient and less stable. The coupling is transferred through market forces, and the price of food is volatile. In India, the coupling is perhaps stabilized by a religion that protects cows.

10.3 ▪ Sustainability

A definition of sustainability often quoted is from the Bruntland Commission that convened in 1987: "development that meets the needs of the present without compromising the ability of future generations to meet their own needs." How would this definition mesh with our "systems approach?" We might consider whether the real part, a, of the eigenvalue, λ, is positive or negative. If it is positive, then e^{at} will grow exponentially. Since both food and biomass crops express a need, and supplies in the form of arable land are finite, we are left to ponder how future generations' needs could be meet in a system where some term is growing exponentially. This argument is the kernel of that advanced by Malthus regarding population growth and food supply. The argument asserts that there is a global carrying capacity for the earth (or any sustainable system) that eventually puts a bound an exponential growth, whether it is population, energy demand, or Gross Domestic Product. It often seems nearly impossible to define that carrying capacity. In a market driven system, the corresponding finitude is introduced by finite resources, the money supply, and the disqualification of an infinite price. Things that cost too much are simply not on the market, and prices that are too high or too low for food and fuel are simply not permissible and certainly not sustainable. The system just falls apart.

The concept of a stable equilibrium for a system is then close to the idea of sustainability and signals a situation where the system doesn't fall apart as $t \to \infty$. This allows that future generations will still be able to meet their needs.

You might think that I am a reductionist offering a rigid, mathematical definition to the very widely used word. But I have no intention of going through the various uses of "sustainable development," "sustainable practice," and "sustainable environment" and measuring them against a "correct" mathematical definition. Instead, I hope to uncover richer meanings that will enhance the discourse and expand the possibility of solutions to the energy problem for the long haul. The American abstract artist Mark Rothko described developments in the early 1940s with the following words: "We are for the large shape because it has the impact of the unequivocal. We wish to reassert the picture plane. We are for flat forms because they destroy illusion and reveal truth" [10, p. 54]. The formal symbols and mathematics at best reassert the frame of the sustainability discussion.

I admit to being somewhat frustrated by not being able to tell you what α, β, γ, and δ actually are. Perhaps you also wish for a book to look up the answer. The field of *integrated assessment* has been developing modeling frameworks from macroeconomic theories to help plan climate change mitigation strategies. The work of Jae Edmonds and others is exemplary in this regard [50]. So how can we advance beyond the discourse and theory to practice and act toward a sustainable energy future?

10.4 ▪ Toward Action

My first suggestion is that we look at the numbers. We need numbers to establish the initial conditions. We cannot do much without knowing this. What is the local carbon footprint and energy demand, and what are the current local climate conditions?

My second suggestion is that we argue about what kinds of solutions are acceptable. This is like establishing the criteria for stability—for what is a permissible solution and what is not. How much volatility and local vulnerability are permissible? Are we willing to take large or small risks in planning our energy infrastructure, and at what cost to health and well being?

My third suggestion is that we figure out how to change the rules of the system. The system matrix will never be adequate to describe our world, and we can change the parameters, even change many of the basic assumptions, by inventing new technologies or new ways to thrive. We should be willing to consider a new system if the current one seems to be falling apart.

Chapter 11

Climate Arrhythmia and Biopolitics

11.1 ▪ Introduction

At the heart of the question of what we should do about climate change is a view of what the climate system is, what is wrong with it, and how we are to play a role in changing it. This question is outside the normal scientific discourse, so you will forgive me for being somewhat more cautious in venturing some observations and comments. This lecture will make a partial sketch of the landscape of this metadiscourse so that our places and roles can be noted as if from the outside. It is easy to be drawn into this discussion because of proposals for geoengineering and proposals for a sustainable energy infrastructure and an economy not based on fossil fuel. We are engineering the planet, whether we like to admit it or not, and these proposals require serious evaluation for their effectiveness, economic feasibility, and impact on environment and society. But what I have in mind here is to step back even further. Here we are in this classroom, a place removed from decision making but open to varieties of discussion and inquiry, so I propose that we look at the way we look at this, that we think about how we think about climate change.

11.2 ▪ Climate Arrhythmia

The word arrhythmia is used in a systems view of the body and the healthy functioning of the heart. The system, the heart in this case, has a rhythm, an established pulse or signature, that is not behaving in a regular fashion. The signature of a healthy heart is read from an electrocardiograph to identify the sinusoidal patterns of the atrial node. The pumping action of the heart moves blood through the four chambers, the left and right atria and the left and right ventricles. The atria are the receiving chambers, and the ventricles are the ejecting chambers. Oxygenated blood arrives from the lungs in the left atria, and the left ventricle pumps this blood into the aorta. The right atrium receives blood from the veins, and the right ventricle pumps this blood to the lungs. The signal of this pumping action is triggered at the sinoatrial node (SA), high in the right atrium, and an electrical-chemical pulse spreads to the left atrium. The electrical signals converge at the atrioventricular node (AV), where a delay allows the atria to empty before the rapid ventricular signal is sent. Arrhythmia typically involves a disruption of the SA node trigger or the conduction pathways in and out of the AV node. An arrhythmia is an unusual disruption of the regular rhythm, usually caused by the loss of the triggering signals, resulting in an inefficient pumping of the heart.

Thinking of the climate system as a living, breathing thing is an obvious metaphor for climatologists who have been on a path to discover and explain the climate system's natural rhythms and variability. There are the natural, healthy rhythms associated with the diurnal cycle of daytime heating and night time cooling and the seasonal changes resulting from the yearly orbital progression of the earth around the sun which changes the solar angle and intensity. These provide the pumping action that distributes life-giving fresh water around the globe. Many other natural rhythms have been discovered and continue to be discovered from more refined measurements and the collection of ocean and weather records over decades and centuries. The periodic build-up and release of heat from the Pacific ocean warm pool off Indonesia cause the Walker circulation in the Pacific to fluctuate and sends us into El Nino or La Nina years. Similar oscillations, the Pacific Decadal Oscillation (PDO) and the North Atlantic Oscillation (NAO), have recently been discovered and are part of the natural rhythms of the coupled atmosphere-ocean system. Long term natural rhythms are also associated with the Milankovich cycles that arise from slight changes in the earth's orbital tilt and are thought to cause periodic ice ages and interglacial warm periods. The cycles and oscillations are likened to the pumping of the heart, and the photosynthetic activity of plants and biota are likened to a seasonal breathing of a global ecosystem lung.

When the climate exhibits an arrhythmia, a disruption of the regular rhythms and patterns, the metaphor carries us to consideration of this as unnatural and unhealthy. Something is wrong. An episode of unusual rhythms might signal that the climate system is getting ready to move to a different base equilibrium state, a new organization of the circulation patterns with different beats and cycles, and a new climate with different weather patterns and distributions of rain, different hot and cool zones, and different intensities of storms. Arrhythmia could signal catastrophic impacts—hence the feeling of panic that these discussions often engender. The use of this metaphor is a way to use our common sense notions of health to explain climate science and the effect that human activity might be having on our natural environment.

11.3 ▪ Reflecting Inward

One of the ways that environmental discourse integrates the scientific and the common sense views of climate is through the construction of myth. In the case of climate science, the explicit participation of scientists like James Lovelock in presenting the Gaia hypothesis is the acknowledgment of the inadequacy of scientific language and quantitative estimates to communicate what is happening and how serious it is. Lovelock anthropomorphizes the climate system with a mother nature figure. The climate system, as a complex set of interactions and feedbacks, is likened to the ancient Greek earth goddess, Gaia. With Gaia, a self-correcting and stabilizing process is identified with past, present, and future quasi-equilibria of the system and in particular with the biosphere's response to climate change. Since the invention of photosynthesis by plants led to a build-up of oxygen in our atmosphere, the global ecosystem regulates and sustains itself like a living, individual, biological organism. Any substantial change to the environment can be expected to provoke a response. For example, as the CO_2 level increases the level of the ocean, algal blooms will increase and ocean production of dimethal sulfide will increase, producing more atmospheric aerosols and controlling the warming of the earth's surface.

In the Greek mythology, Gaia provides the nurturing environment for humans. But for many modern day users of this metaphor, humans are the germs, the disruptive entities that the bodily immune system must somehow keep in check. The unabated growth of fossil fuel use is likened to an addiction and human population rise to the growth of a cancer. That Gaia's response may be angry puts human survival squarely on the table and sets nature at odds with our future. These are strong mythological and motivational images.

A common sense view isn't necessarily the thing that makes the most sense; often it is the thing that is easiest. For example, there is the notion that what we are observing in climate change is somehow inevitable and that even if it is a problem, we have always muddled through, so not to worry. That is very much what we want to believe, and since it requires essentially no changes or commitments on our part, it is the easiest view to adopt. Could it be the right view? Could it be that other people, besides us, will solve the problems and have solved them in the past, and that we are simply along for the ride? No one can fault us for just trying to get by. After all, life is tough, and we cannot be expected to worry about anyone but ourselves. The Gaia hypothesis assaults this view by exposing a darker side, a potential end to our nurturing environment.

What is being sought in the discourse is a a way to think about the climate problem and about ourselves. Leo Marx examines the metaphor of the machine disturbing idyllic nature in the classic *The Machine in the Garden*. His study of the role and relationship of nature in literature beginning with Hawthorne's *Sleepy Hollow* finds an internal reflection of human nature in the confrontation of nature by industrial progress. "The primary subject is the contrast between two conditions of consciousness…the chief concern is the landscape of the psyche" [30, p. 28]. The industrial revolution brings a new type of event and "a sense of history as an unpredictable, irreversible sequence of unique events" [30, p. 31]. What is developed through literature and philosophy is an uncommon sense that asserts that "mind and nature are in essence one" [30, p. 67]. The climate arrhythmia we are now experiencing calls for action. The patient needs urgent care and therapy guided by an uncommon knowledge. And the patient is just as much us as it is nature.

11.4 ▪ Biopolitics

In the broader philosophical discussion, the theme is decidedly similar for the various crises of civilization in emphasizing a therapeutic role for intellectual effort and social reform. Control of the conceptual apparatus is often seen as the key to finding solutions for deep problems and instilling new behaviors. The regulation of behavior, a function of both religion and government, has expanded over the recent past from morality and international conflicts to economics and lifestyle and now to the environment and species survival. Michel Foucault coined the term "biopolitics" to describe this shift and the seminal change he observed in the practice of governance. What he calls governance in the broadest sense has seen a very much expanded agenda in comparison to the past centuries' nation states and the authority of the church. We are no longer followers of a monarchy or an overriding religious authority. There has been a change in the exercise of power that seeks to influence much broader areas of behavior. How present behaviors and energy technologies could change over the next 50 years to produce a reduction in greenhouse gas emissions by nearly 70% over the year 2000 levels is the objective of this exercise of power. Our practices of governance are performed on a different stage with different assumed boundaries and modes of presentation than before. The manipulation and control of global photosynthesis is now our direct concern and the concern of our governments.

How do we imagine ourselves living in a sustainable world? The ideal of an agrarian society informed our predecessors. But in an increasingly global economy with rapid growth of cities, this ideal requires a new imagining. Wendell Berry is one writer who has sought to explore this in his term "imagination in place."

> when one passes from an abstract order to the daily life and work of one's own farm, one passes from a relative simplicity into a complexity that is irreducible except by disaster and ultimately is incomprehensible. It is the complexity of the life of a place uncompromisingly itself, which is at the same time the life of the world, of all creation [3, p. 48].

His book, *The Unsettling of America*, documents the decline of the family farm in America and the move away from the agrarian ideal. Many of his other books deal with a broader issue. My interest in his writing is not the nostalgic sense of loss that pervades our unsettling, but his advocacy of "place" as a therapuetic solution to a crisis that he sees primarily in terms of our imagination.

Our inability to imagine and manage something as complex as the global environment becomes apparent as we confront our ignorance over how the climate system works and how little success we have in regulating our own behaviors. Polymathic ignorance is what I am likely accused of as "the ignorance of people who know all about long term consequences in the future" [3, p. 55]. The use of computers and climate models adds to the notion that we are not doing our own imagining but letting the machines plan for us, as if the future were a scheduling program, an action plan to be managed. Stewart Brand writes about the political agenda of the Green movement after the IPCC AR4 reports of 2007 that it was

> suddenly outdated - too negative, too tradition bound, too specialized, too politically one sided for the scale of the climate problem...Accustomed to saving natural systems from civilization, Greens now have the unfamiliar task of saving civilization from a natural system - climate dynamics [6, p. 209].

The year 2007 marks a shift in awareness and a shift in the view of who, how, and when we will deal with the "climate problem." Brand states that businesses and engineers are currently at the forefront of innovation with LEED (Leadership in Energy and Environmental Design) buildings and new energy technologies, and that there is a lesser role for the traditional environmental advocates. This signals an acknowledgment that the problem has been recognized and it is time to do something about it.

Foucault saw that the art of governance was extending to the management of life's rhythms. The practice of this management constitutes a new form of being-in-the-world, a new cultural identity. *This is what we are trying to imagine.* This is the crisis of the post-modern era after the modernist dream of better living through science. The modernist legacy is a broad based material prosperity but also systematic genocide and the development of self-annihilation capability with nuclear, biologic, and now genetic technologies. What we do about climate change over the next several decades will be a practice of "governmental reason" [19, p. 22] that will define biopolitics and the way we can imagine ourselves and our future.

In his essay *Unbehagen in der Nature* [51] Slavoj Zizek writes

> Although concerns about global warming explode from time to time and are gaining more and more scientific credibility, ecology as an organized socio-political movement has to a large degree disappeared [51, p. 439].

This is evident even at professional meetings as the observational environmentalists give way to the "systems approach" modelers who dominate the global climate change science. The scientific method of documenting impacts of climate change on ecosystems has lead to a distasteful atmosphere of fear and an apocalyptic outlook toward the collapse of ecosystems.

> This ecology of fear has every chance of developing into the predominant form of ideology of global capitalism, a new opium for the masses replacing declining religion; it takes over the old religions' fundamental function, that of having an unquestionable authority which can impose limits...This is why, although ecologists are all the time demanding that we radically change our way of life, underlying this demand is its opposite, a deep distrust of change, of development, of progress; every radical change can have the unintended consequence of triggering a catastrophe [51, p. 439].

Nature has taken on a dominating role to which everything must be related: green products, green practices, green ideals for a sustainable future. Practical political goals and power are exercised with an ideal of Nature in mind. And strategies exhibit a fundamental tension that this religious dedication to a changeless Nature embody.

Zizek's analysis and suggestion is that we change our cultural, conceptual framework.

> The problem is thus that we rely neither on the scientific mind nor on our common sense – they both mutually reinforce each others' blindness. The scientific mind advocates a cold objective appraisal of dangers and risks involved where no such appraisal is really possible, while common sense finds it hard to accept that a catastrophe can really occur. The difficult ethical task is to 'un-learn' the most basic coordinates of our immersion into our life-world [51, p. 445].

In a thought provoking and somewhat shocking terminology, Wendell Berry calls this "un-learning" and the path it entails "the way of ignorance." He casts this term toward a different kind of knowing, not as a matter of knowing nothing, but as an alternative to our heavy reliance on knowledge to decide how we should behave ethically. This is not an antiscientific view, though ignorance seems to be held up as a virtue in his choice of title. For ethical choices there is a long tradition of wisdom literature that does not collect or espouse knowledge. In fact, he says this path is the way of neighborly love, kindness, caution, care, appropriate scale, thrift, good work, and right livelihood. To contrast this way with the path we currently follow Berry says that "creatures who have armed themselves with the power of limitless destruction should not be following any way laid out by their limited knowledge and their unseemly pride in it" [3, p. ix].

Defining a way is more than making a set of decisions. If it were only this, then perhaps accumulated knowledge would suffice. Knowledege may be less useful in helping us stick with the decisions and providing a source of stamina and patience over years or centuries of living. Speaking of a way suggests that good decisions imply qualities that must be discovered over a course of time. It is a decision that is made over and over again, each time with the nuance of a life of experiences and refined knowledge. A way is held in the knowledge of source and destination, and one travels with grace or not.

It may sound like these are the same old coordinates being expressed without the authority of church or state but mapped back to coordinates defined by our religious traditions. Berry would acknowledge this, I think, without hesitation while denying any particular doctrinary formula. Indeed, these coordinates are operative in much thinking about the climate system whether they derive from the explicit religious idea of stewardship or from ancient agricultural traditions. One way to look at this is that the religious traditions grew up to enforce and reinforce the social relations required to make agriculture possible. But Berry, as well as other writers (e.g., Barry Lopez and David Orr), appear to be seeking a basis for things common in a unity of place that can be reinvented based on personal and local history and faith. My father, the historian Richard Drake, has used an analysis based on identification of land, agriculture, and discipline to understand the cultural development of Appalachia. For many mountain dwellers "there is a place for a viable, yeomanesque-style of life that is attractive to those unwilling or unable to join the mainstream's affluence" [16, p. 246]. This and the agrarian lifestyle are extensions of the Jeffersonian ideal that propelled Americans westward and captured the tenacious hope of many Appalachians of having land of one's own to be passed from generation to generation.

As a scientist and (now) educator, my natural inclination is to recommend that we augment common sense with analytics and develop a contextual information presence in our daily life. We could use the emerging ubiquitous computing resources to continually "run the numbers" relating actions to consequences and make that information available to augment our imagination and judgment. This might be called the "enlightened design" solution to the ethical problem of remapping our life-coordinates. Humans could become ethical by standardizing our interface with the environment and reducing the big-N-Nature to quantifiable rules and formulas.

Another direction could be called the pursuit of "compassionate discovery." While the enlightened design solution redefines and limits how we imagine ourselves and nature, the compassionate discovery direction focuses and opens new avenues of understanding nature. A new logic for the observation and study of nature would need to be developed. A part of this logic might make sense of interspecies understanding, recognizing the face that stares out from wildness and extending the sphere of power and responsibility beyond a notion of stewardship. The scientific attitude of awe might turn into reverence like that exhibited by Barry Lopez as he respectfully removes dead animals from the sides of roads. In a scientific challenge of our traditional truths, we would hope to discover a new earth inside or beside the old one, a new concept of nature that embraces the multitude of living things with principles and laws that define a sustainable place for us.

The trajectory we are currently following is much closer to the notion of enlightened design. We may even think of ourselves as Brand says, "We are as gods and have to get good at it." This solution, though static, is highly unstable, and many would argue that it is what got us into our current predicament. In contrast, the pole of discovery is fundamentally stable. Unfortunately, it is much further removed and less affective of our current decisions. The climate is sending signals symptomatic of a deeper problem requiring therapy, healing, and change to the very way we look at ourselves. The metaphor of climate arrhythmia suggests a revaluation of place and discovery of different designs for life. The outlines of the climate change discourse draw on modern and postmodern notions of the way we know our world and our place in it. The logic of our choices requires a ground for the ethical in a yet unapproachable ineffable relationship with the living, breathing environment and requires that we lean upon our ignorance as well as our most advanced science. Charting a healthy, ethical course between the two poles of design and discovery is seen as the way to address the climate problem and imagine the path of a developing biopolitics.

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