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# Preface

This book is intended to foster the interchange of ideas between partial differential equations (PDEs) and dynamical systems. It is designed for beginning graduate students and researchers in the area to obtain an overview of the myriad possibilities to apply dynamical systems techniques to PDEs as well as to highlight the impact of PDE methods on dynamical systems. In recent years, the two fields have seemingly drifted further apart in teaching than may seem strictly necessary. A first sequence of introductory PDE courses tends to focus on the existence and regularity aspects while dynamical systems courses tend to focus on applications to ordinary differential equations (ODEs) or discrete-time iterated maps. Although it would be excellent if students would take all possible courses, this is rarely practical. Hence, one often encounters PDE practitioners well versed in functional-analytic inequality-based arguments or those more comfortable with a geometry-based dynamics approach. The combination is relatively rare. This book aims to change this situation by highlighting how the different approaches are intertwined. A natural alternative title for the book could have been

*A Concise Introduction to the Interactions between Dynamical Systems and Partial Differential Equations*

but I found the title too lengthy in the end; nevertheless, it still describes the intentions quite well. Since many prefaces of mathematics books tend to be slightly dull, I decided to try to summarize principles set out for this book in a different format:

- **Applications:** Although in such a short book no real-world cutting-edge applications can be treated in depth, the generic techniques are designed to be very applicable for broad classes of PDEs. My strategy was to cover PDEs, which have turned out to be incredibly applicable in the past. Of course, the additional assumption is that it pays off to study *classics* to be prepared for the future.
- **Background:** As background knowledge a first course in PDEs is helpful, e.g., one based upon the introductory parts of [186, 472]. Some basic knowledge of ODEs [24, 270] is naturally desirable as well. Further additional background is often elementary, and one may look up the required results along the way.
- **Choice:** The references show that there is certainly *a lot* of choice for an introduction to PDE dynamics. I have tried to omit topics which place *too much strain on more specialized background*.
- **Design:** The focus of the book is to introduce key concepts and important analysis strategies at the interface between dynamical systems and PDE. The idea is to condense, contrast, and combine different viewpoints to maximize the preparation for research, but still keep the size of the book relatively short.

- **Examples:** All results are illustrated by concrete PDE examples. This approach automatically introduces a key suite of benchmark PDEs, which have guided the development of the general theory.
- **Figures:** Figures are kept deliberately simple and noncomputational in the main part of the text; see the appendices for basic computational tools. The approach allows one to directly utilize the book for blackboard/classroom use.
- **Genre:** A very broad selection of topics is presented to allow beginners an initial overview of the potential directions. However, the book is obviously *not* a monograph.
- **Highlights:** Since the book does not even attempt to be exhaustive in its area, several highlight results have been selected. Certainly, this selection is highly debatable, but it does exemplify the rapid growth of the field over the last couple of decades.
- **Ideas:** It is often necessary to learn a toolbox of “standard” tricks, proof strategies, or computations in a new area. This book tries to introduce as much as possible from this standard toolbox. However, the high density of ideas necessitates that we demonstrate each new technique often only once, or at most twice. So one has to adapt the lecturing/reading pace accordingly.
- **Jokes:** Albeit (hopefully!) being “fun-to-read” from a scientific perspective, jokes have been omitted as they almost surely do not transmit well in mathematics writing; and the last sentence provides a relevant example for the keen probabilist.
- **Kickoff:** As an author, I would be particularly pleased if this book would form not the end of a journey for you. Rather the opposite outcome would be desirable. The concept is to push you towards deeper questions at the very vibrant interface of dynamics and PDEs.
- **Lectures:** Each chapter corresponds approximately to a lecture of ninety minutes. Some chapters come as packages of 2–4 connected units; the overall structure is outlined in the section “Course Design” following the Preface. The style allows lecturers to relatively freely combine topics into a course that fits their needs and interfaces well with other courses taught within the curriculum.
- **Multipurpose:** Each chapter could also be the basis of a seminar talk. Furthermore, the book is suitable for self-study to get an overview of the area. For beginning graduate or advanced undergraduate students, it is also an excellent step to actually see all the basic mathematics from the first few semesters *in action*.
- **Notation:** The notation has been kept simple, in the hope of avoiding confusion with too many symbols and variable names. In particular, one objective was to make the notation relatively easy to use on the blackboard, and to have a fully consistent scheme for the main global objects, yet potentially reuse variable names for auxiliary local objects.
- **Omissions:** There have been many cuts. I shall just mention four cases to illustrate the problem: (1) Focusing more on fluid dynamics was tempting as it is a topic deeply connected to the development of PDEs and dynamical systems.

(2) Integral/nonlocal PDEs would have been an interesting field to explore. (3) PDEs arising in geometric analysis, e.g., in the context of Ricci flow or general relativity, were also an intriguing option. (4) There are many links to stochastic problems, which I have—from a personal perspective extremely reluctantly—decided to omit.

- **Proofs:** As many proofs as possible are included, particularly if there are key ideas and new strategies contained within a proof. However, in a topic as broad as PDE dynamics, and taking into account the introductory approach, we have to refer to specialized books for very long technical arguments. However, if a proof is omitted, then a precise reference is provided, where a proof can be found.
- **Questions:** For each chapter, three exercises are provided. There are many more potential exercises that could have been selected. The idea is that for a lecture course or self-study, it is usually very unclear which of the many exercises one should try. Here the solution is simple: just do all of them if a chapter is covered in a course or relevant for your research. Most of the exercises are not difficult and are designed to just keep you thinking about the material.
- **References:** At the end of each chapter, some references are mentioned to spark interest in further reading. The reference list is certainly incomplete. Since this is an introduction to PDE dynamics, I decided to have a bias in this book towards citing other, generally more monographic texts or review papers for certain subclasses of PDE problems.
- **Style:** The format is based around relatively short units/chapters. The principle upon which this assumption is based centers around the unavoidable fact that readers may want to only delve into a subset of all topics. Although this means introducing some notations again, I feel this tradeoff towards modularity is well worth it.
- **Termination:** Since I simply do not need the letters U–Z to convey what I wanted to say about the structure of this book, I shall simply take the liberty to terminate this part of the introduction rather unexpectedly.

In addition to the previous remarks, I have now the pleasure to thank Luca Arcidiacono, Tobias Böhle, Marcel Braukhoff, Paul Carter, Annalisa Iuorio, Tobias Jawecki, Chris Münch, Anne Pein, Stefan Portisch, Pedro Aceves Sanchez, Elisabeth Schiessler, Lara Trussardi, Hannes Uecker, and Julian Westermeier for alerting me to several mistakes and misprints in previous versions of this book. Parts of this book have been used for a course at Vienna University of Technology in fall/winter 2014/2015 and for a course at Technical University of Munich (TUM) in fall/winter 2016/2017 and I would like to thank all students for bearing with me while I tried out this relatively new format. In fact, providing a broad, yet concise, introductory account for the next generation of PDE dynamics researchers was a key motivation to write the book in the first place.

Regarding the figures in this book, I am indebted to Andreas Burkhart, who converted my hand-drawn scribbles into sustainable vector graphics and postscript images.

Although it is clear to most readers, despite the support of colleagues as well as the editorial staff at SIAM, let me emphasize that I am entirely responsible for all remaining errors and inaccuracies remaining in this version of the book. I am going to maintain a web page to provide improvements and bug fixes, so please report any errors or inaccuracies you find to me via email.

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