Additional Exercises
for *Introduction to Nonlinear Optimization*

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Chapter 1 - Mathematical Preliminaries

1.1 Let $S \subseteq \mathbb{R}^n$.

(a) Suppose that T is an open set satisfying $T \subseteq S$. Prove that $T \subseteq \text{int}(S)$.

(b) Prove that the complement of $\text{int}(S)$ is the closure of the complement of S.

(c) Do S and $\text{cl}(S)$ always have the same interiors?

1.2 For any $\mathbf{x} \in \mathbb{R}^n$ and any nonzero vector $\mathbf{d} \in \mathbb{R}^n$, compute the directional derivative $f'(\mathbf{x};\mathbf{d})$ of

$$f(\mathbf{x}) = \left| \|\mathbf{x}-\mathbf{a}\|_2 - \delta \right| + \max\{\mathbf{c}_1^T\mathbf{x}+\beta_1, \mathbf{c}_2^T\mathbf{x}+\beta_2\},$$

where $\mathbf{a}, \mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^n$ and $\delta \in \mathbb{R}_{++}, \beta_1, \beta_2 \in \mathbb{R}$.

Chapter 2 - Optimality Conditions for Unconstrained Optimization

2.1 For each of the following matrices determine, without computing eigenvalues, the interval of α for which they are positive definite/negative definite/positive semidefinite/negative semidefinite/indefinite:

(a) $\mathbf{B}_\alpha = \begin{pmatrix} -1 & \alpha & -1 \\ \alpha & -4 & \alpha \\ -1 & \alpha & -1 \end{pmatrix}$.

(b) $\mathbf{E}_\alpha = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 2 & \alpha & 0 \\ 0 & \alpha & 2 & \alpha \\ 0 & 0 & \alpha & 1 \end{pmatrix}$.

2.2 Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Assume that there exist two indices $i \neq j$ for which $A_{jj}, A_{ij} \neq 0$ and $A_{ii} = 0$. Prove that $\mathbf{A}$ is indefinite.

2.3 Let $f(x_1, x_2) = x_1^2 - 2x_1x_2^2 + \frac{1}{2}x_2^4$.

(a) Is the function f coercive? explain your answer.

(b) Find the stationary points of f and classify them (strict/nonstrict local/global minimum/maximum or a saddle point).

2.4 Consider the function $f(x, y) = x^2 - x^2y^2 + y^4$. Find all the stationary points of f and classify them (strict/non-strict, local/global, minimum/maximum or a saddle point).

Chapter 3 - Least Squares

3.1 Let $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{L} \in \mathbb{R}^{p\times n}$ and $\lambda \in \mathbb{R}_{++}$. Consider the function

$$f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{L}\mathbf{x}\|_1.$$

(a) Show that if f is coercive, then $\text{Null}(\mathbf{A}) \cap \text{Null}(\mathbf{L}) = \{\mathbf{0}\}$.

(b) Show that the contrary also holds, i.e., if $\text{Null}(\mathbf{A}) \cap \text{Null}(\mathbf{L}) = \{\mathbf{0}\}$ then f is coercive.

Chapter 6 - Convex Sets

6.1 Show that the following set is not convex:

$$S = \{\mathbf{x} \in \mathbb{R}^2 : 3x_1^2 + x_2^2 + 4x_1x_2 - x_1 + 4x_2 \le 10\}.$$

6.2 (a) Prove that the extreme points of $\Delta_n = \{\mathbf{x} : \sum_{i=1}^n x_i = 1, x_i \ge 0\}$ are given by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

(b) For each of the following sets specify the corresponding extreme points:

(i) $\{\mathbf{x} : \mathbf{e}^T\mathbf{x} \le 1, \mathbf{x} \ge \mathbf{0}\}$.

(ii) $B[\mathbf{c}, r]$ where $\mathbf{c} \in \mathbb{R}^n$ and $r > 0$.

(iii) $\{(x_1, x_2)^T : 9x_1^2 + 16x_2^2 + 24x_1x_2 - 6x_1 - 8x_2 + 1 \le 0, x_1 \ge 0, x_2 \ge 0\}$.

Chapter 7 - Convex Functions

7.1 Find the optimal solution of the problem

$$\begin{array}{ll} \max_{\mathbf{x}\in\mathbb{R}^3} & 2x_1^2 + x_2^2 - x_3^2 + 2x_1 - 3x_2 + 4x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

7.2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, and let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix. Consider the function h defined as follows:

$$h(\mathbf{y}) = \min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{y}\},$$

where we assume that $h(\mathbf{y}) > -\infty$ for all $\mathbf{y}$. Show that h is convex.

Chapter 8 - Convex Optimization

8.1 Consider the problem

$$\begin{array}{ll} \min & \max\{|x_1 - x_2|, |x_2 - x_3|, |x_1 - x_3|\} + x_1^2 + 2x_2^2 + 3x_3^2 - 2x_2 x_3 \\ \text{s.t.} & (4x_1^2 + 6x_2^2 - 2x_1 x_2 + 1)^4 + \frac{x_3^2}{x_1 + x_2} \leq 150 \\ & x_1 + x_2 \geq 1 \end{array}$$

(a) Show that it is convex.

(b) Write a CVX code that solves it.

(c) Write down the optimal solution (by running CVX).

8.2 Consider the following convex optimization problem:

$$\begin{array}{ll} \min & \sqrt{2x_1^2 + 4x_1 x_2 + 3x_2^2 + 1} + 7 \\ \text{s.t.} & ((x_1^2 + x_2^2)^2 + 1)^2 \leq 100 x_1 \\ & \frac{x_1^2 + 4x_2^2 + 4x_1 x_2}{2x_1 + x_2 + x_3} \leq 10 \\ & 1 \leq x_1, x_2, x_3 \leq 10. \end{array}$$

(a) Prove that the problem is convex.

(b) Write a CVX code for solving the problem.

8.3 Prove that the following problem is convex in the sense that it consists of minimizing a convex function over a convex feasible set.

$$\begin{array}{ll} \min & \log\left(e^{x_1 - x_2} + e^{x_2 + x_3}\right) \\ \text{s.t.} & x_1^2 + x_2^2 + 2x_3^2 + 2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3 \leq 1, \\ & (x_1 + x_2 + 2x_3)(2x_1 + 4x_2 + x_3)(x_1 + x_2 + x_3) \geq 1, \\ & e^{e^{x_1}} + [x_2]_+^3 \leq 7, \\ & x_1, x_2, x_3 \geq \frac{1}{10}. \end{array}$$

8.4 Consider the problem

$$\begin{array}{lll} & \min & a x_1^2 + b x_2^2 + c x_1 x_2 \\ (\mathrm{Q}) & \text{s.t.} & 1 \leq x_1 \leq 2, \\ & & 0 \leq x_2 \leq x_1. \end{array}$$

where $a, b, c \in \mathbb{R}$.

(a) Prove that there exists a minimizer for problem (Q).

(b) Prove that if $a < 0, b < 0$ and $c^2 - 4ab \leq 0$, then the optimal value of problem (Q) is

$$4 \min\{a, a + b + c\}.$$

(c) Prove that if $a > 0, b > 0$ and $c^2 > 4ab$, then problem (Q) has a unique solution.

8.5 Consider the optimization problem

$$
\text{(P)} \quad \begin{array}{ll} \min & \sqrt{x_1^2 + 4x_1x_2 + 5x_2^2 + 2x_1 + 6x_2 + 5} - \frac{x_2}{x_2+1} \\ \text{s.t.} & (|x_1 - 1| + |x_2 - 1|)^2 + \frac{x_1^4 - x_2^2}{x_1^2 + x_2} \leq 7 \\ & x_2 \geq 1. \end{array}
$$

(a) Prove that $x_1^2 + 4x_1x_2 + 5x_2^2 + 2x_1 + 6x_2 + 5 \geq 0$ for any x_1, x_2.

(b) Prove that problem (P) is convex.

(c) Write a CVX code that solves the problem.

8.6 Consider the problem

$$
\text{(P)} \quad \max\{g(\mathbf{y}) : f_1(\mathbf{y}) \leq 0, f_2(\mathbf{y}) \leq 0\},
$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is concave and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex. Assume that the problem $\max\{g(\mathbf{y}) : f_1(\mathbf{y}) \leq 0\}$ has a unique solution $\tilde{\mathbf{y}}$. Let Y^* be the optimal set of problem (P). Prove that exactly one of the following two options holds:

(i) $f_2(\tilde{\mathbf{y}}) \leq 0$ and in this case $Y^* = \{\tilde{\mathbf{y}}\}$.

(ii) $f_2(\tilde{\mathbf{y}}) > 0$ and in this case $Y^* = \text{argmax}\{g(\mathbf{y}) : f_1(\mathbf{y}) \leq 0, f_2(\mathbf{y}) = 0\}$.

8.7 Show that the following optimization problem can be cast as a convex optimization problem and write a CVX code for solving it.

$$
\begin{array}{ll} \min & \frac{4x_1^2 + 2x_1x_2 + 5x_2^2}{x_1 + x_2} + (x_1^2 + x_2^2 + 1)^2 \\ \text{s.t.} & \frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} \geq 1 \\ & x_1^2 \leq x_2\sqrt{2x_1 + 3x_2} \\ & x_1, x_2 \geq 1. \end{array}
$$

8.8 Show that the following is a convex optimization problem and write a CVX code for solving it.

$$
\begin{array}{ll} \min & x_1^2 + (4x_1 + 5x_2)^2 - (3x_1 + 4x_2 + 1)^2 \\ \text{s.t.} & \sqrt{x_1^2 + x_2^2 + 1} \leq \sqrt{x_2} \\ & \frac{x_1^2 - x_2^2}{x_2} \leq \min\{x_2 - |x_1 + 3x_2|, 7\} \\ & x_2 \geq 1. \end{array}
$$

Chapter 9 - Optimization over a Convex Set

9.1 Consider the set

$$
\text{Box}[\boldsymbol{\ell}, \mathbf{u}] \equiv \{\mathbf{x} \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i, i = 1, 2, \ldots, n\}
$$

where $\boldsymbol{\ell}, \mathbf{u} \in \mathbb{R}^n$ are given vectors that satisfy $\boldsymbol{\ell} \le \mathbf{u}$. Consider the minimization problem

$$\text{(P)} \quad \min\{f(\mathbf{x}) : \mathbf{x} \in \text{Box}[\boldsymbol{\ell}, \mathbf{u}]\},$$

where f is continuously differentiable function over $\text{Box}[\boldsymbol{\ell}, \mathbf{u}]$. Prove that $\mathbf{x}^* \in \text{Box}[\boldsymbol{\ell}, \mathbf{u}]$ is a stationarity point of (P) if and only if

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) \begin{cases} = 0, & l_i < x_i^* < u_i, \\ \le 0, & x_i^* = u_i, \\ \ge 0, & x_i^* = l_i. \end{cases}$$

9.2 In the "source localization problem"[1] we are given m locations of sensors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and approximate distances between the sensors and an unknown "source" located at $\mathbf{x} \in \mathbb{R}^n$:

$$d_i \approx \|\mathbf{x} - \mathbf{a}_i\|_2.$$

The problem is to find/estimate $\mathbf{x}$ given the locations $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ and the approximate distances $d_1, d_2, \ldots, d_m$. The following natural formulation of the problem as a minimization problem was introduced in Exercise 4.5:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \sum_{i=1}^m (\|\mathbf{x} - \mathbf{a}_i\|_2 - d_i)^2 \right\}. \tag{SL}$$

(a) Show that the problem given by (SL) is equivalent to the problem

$$\begin{array}{ll} \min & f(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m) \equiv \sum_{i=1}^m \|\mathbf{x} - \mathbf{a}_i\|_2^2 - 2d_i \mathbf{u}_i^T(\mathbf{x} - \mathbf{a}_i) + d_i^2 \\ \text{s.t.} & \|\mathbf{u}_i\|_2 \le 1, i = 1, \ldots, m, \\ & \mathbf{x} \in \mathbb{R}^n, \end{array} \tag{SL2}$$

in the sense that $\mathbf{x}$ is an optimal solution of (SL) if and only if there exists $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)$ such that $(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)$ is an optimal solution of (SL2).

(b) Find a Lipschitz constant of the function f.

(c) Consider the two-dimensional problem ($n = 2$) with 5 anchors ($m = 5$) and data generated by the MATLAB commands

```
randn('seed',317);
A=randn(2,5);
x=randn(2,1);
d=sqrt(sum((A-x*ones(1,5)).^2))+0.05*randn(1,5);
d=d';
```

[1] The description of the problem also appears in Exercise 4.5

The columns of the 2×5 matrix $\mathbf{A}$ are the locations of the five sensors, $\mathbf{x}$ is the true location of the source and $\mathbf{d}$ is the vector of noisy measurements between the source and the sensors. Write a MATLAB function that implements the gradient projection algorithm employed on problem (SL2) for the generated data. Use the following step size selection strategies

(i) constant step size.

(ii) backtracking with parameters $s = 1,\ \alpha = 0.5,\ \beta = 0.5$.

Start both methods with the initial vectors $(1000, -500)^T$ and $\mathbf{u}_i = \mathbf{0}$ for all $i = 1, \ldots, m$. Run both algorithms for 100 iterations and compare their performance.

Chapter 10 - Optimality Conditions for Linearly Constrained Problems

10.1 Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove that the following two claims are equivalent.

(A) The system

$$\mathbf{A}\mathbf{x} = \mathbf{0}, \mathbf{x} > \mathbf{0}$$

has no solution.

(B) There exists a vector $\mathbf{y} \in \mathbb{R}^n$ for which $\mathbf{A}^T\mathbf{y} \leq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y}$ is not the zeros vector.

10.2 Consider the problem

$$\text{(Q)} \quad \begin{array}{ll} \min_{\mathbf{x} \in \mathbb{R}^n} & \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{d}^T\mathbf{x} \\ \text{s.t.} & \mathbf{a}_1^T\mathbf{x} \leq b_1, \\ & \mathbf{a}_2^T\mathbf{x} = b_2, \end{array}$$

where $\mathbf{Q} \in \mathbb{R}^{n \times n}$ $(n \geq 3)$ is positive definite, $\mathbf{d}, \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n$ and $b_1, b_2 \in \mathbb{R}$. Assume that $\mathbf{a}_1^T\mathbf{Q}^{-1}\mathbf{a}_1 = \mathbf{a}_2^T\mathbf{Q}^{-1}\mathbf{a}_2 = 2, \mathbf{a}_2^T\mathbf{Q}^{-1}\mathbf{a}_1 = 0, \mathbf{a}_1 \neq \mathbf{0}, \mathbf{a}_2 \neq \mathbf{0}$

(a) Are the KKT conditions necessary and sufficient for problem (Q)? explain your answer.

(b) Prove that the problem is feasible.

(c) Write the KKT conditions explicitly.

(d) Find the optimal solution of the problem.

Chapter 11 - The KKT Conditions

11.1 Consider the problem

$$\begin{array}{ll} \max & x_1^3 + x_2^3 + x_3^3 \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 = 1. \end{array}$$

(a) Is the problem convex?

(b) Prove that all the local maximum points of the problem are also KKT points.

(c) Find all the KKT points of the problem.

(d) Find the optimal solution of the problem.

Chapter 12 - Duality

12.1 Consider the following optimization problem:

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\mathbb{R}^n, t\in\mathbb{R}} & \frac{1}{2}\|\mathbf{x}\|_2^2 + ct \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} + t\mathbf{e} \\ & t \geq 0, \end{array} \tag{P5}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, c \in \mathbb{R}$, and as usual $\mathbf{e}$ is the vector of all ones. Assume in addition that the rows of $\mathbf{A}$ are linearly independent.

(i) Find a dual problem to problem (P5) (do not assign a Lagrange multiplier to the nonnegativity constraint).

(ii) Solve the dual problem obtained in part (i) and find the optimal solution of problem (P5).

12.2 Consider the optimization problem (with the convention that $0 \log 0 = 0$):

$$\begin{array}{ll} \min & \mathbf{a}^T\mathbf{x} + \sum_{i=1}^n x_i \log x_i \\ \text{s.t.} & \mathbf{x} \in \Delta_n, \end{array}$$

where $\mathbf{a} \in \mathbb{R}^n$ and Δ_n is the unit simplex.

(i) Show that the problem cannot have more than one optimal solution.

(ii) Find a dual problem in one dual decision variable.

(iii) Solve the dual problem.

(iv) Find the optimal solution of the primal problem.

12.3 Let $\mathbf{E} \in \mathbb{R}^{k\times n}, \mathbf{f} \in \mathbb{R}^n, \mathbf{a} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p \in \mathbb{R}^n$. Consider the problem

$$\text{(P)} \quad \begin{array}{ll} \min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{z}\in\mathbb{R}^m} & \frac{1}{2}\|\mathbf{E}\mathbf{x}\|_2^2 + \frac{1}{2}\|\mathbf{z}\|_2^2 + \mathbf{f}^T\mathbf{x} + \mathbf{a}^T\mathbf{z} + \sum_{i=1}^p e^{\mathbf{c}_i^T\mathbf{x}} \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}, \\ & \mathbf{z} \geq \mathbf{0}. \end{array}$$

Assume that $\mathbf{E}$ has full column rank.

(a) Show that the objective function is coercive.

(b) Show that if the set $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \le \mathbf{b}\}$ is nonempty, then strong duality holds for problem (P).

(c) Write a dual problem for problem (P).

12.4 Consider the problem

$$\begin{array}{ll} \min_{\mathbf{x},\mathbf{y}\in\mathbb{R}^n} & \sqrt{\|\mathbf{x}\|_2^2 + 4} + \mathbf{a}^T\mathbf{y} + \|\mathbf{x}\|_2 \\ \text{s.t.} & \mathbf{B}\mathbf{x} + \mathbf{C}\mathbf{y} \le \mathbf{d}, \\ & \|\mathbf{y}\|_2 \le 1, \end{array}$$

where $\mathbf{a} \in \mathbb{R}^n, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{m\times n}, \mathbf{d} \in \mathbb{R}^m$.

(a) Show that if $\mathbf{B}\mathbf{B}^T \succ \mathbf{0}$, then strong duality holds for the problem.

(b) Find a dual problem.

12.5 Consider the following convex optimization problem:

$$\text{(P)} \quad \begin{array}{ll} \min & \|\mathbf{A}\mathbf{x} + \mathbf{b}\|_2 + \|\mathbf{L}\mathbf{x}\|_1 + \|\mathbf{M}\mathbf{x}\|_2^2 + \sum_{i=1}^n x_i \log x_i \\ \text{s.t.} & \mathbf{x} \ge \mathbf{0}, \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{L} \in \mathbb{R}^{p\times n}, \mathbf{M} \in \mathbb{R}^{q\times n}$. Write a dual problem of (P). Do not perform any transformations that ruin the convexity of the problem.

12.6 (a) Prove that for any $\mathbf{a} \in \mathbb{R}^n$, the following holds:

$$\min \mathbf{a}^T\mathbf{x} + \|\mathbf{x}\|_\infty = \begin{cases} 0, & \|\mathbf{a}\|_1 \le 1, \\ -\infty, & \text{else.} \end{cases}$$

(b) Consider the following minimization problem:

$$\text{(P)} \quad \begin{array}{ll} \min & \sqrt{\|\mathbf{A}\mathbf{x}\|_2^2 + 1} + \|\mathbf{x}\|_\infty \\ \text{s.t.} & \mathbf{B}\mathbf{x} \le \mathbf{c}, \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{d\times n}, \mathbf{B} \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^m$. Assume that the problem is feasible. Find a dual problem to (P). Do not perform any transformations that might ruin the convexity of the problem.

12.7 Consider the problem

$$\text{(G)} \quad \begin{array}{ll} \min & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} \\ \text{s.t.} & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{c}^T\mathbf{x} + d \le 0, \end{array}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is positive definite, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

(a) Under which (explicit) condition on the data $(\mathbf{Q}, \mathbf{b}, \mathbf{c}, d)$ is the problem feasible?

(b) Under which (explicit) condition on the data $(\mathbf{Q}, \mathbf{b}, \mathbf{c}, d)$ does strong duality hold?

(c) Find a dual problem to (G) in one variable.

(d) Assume that $\mathbf{Q} = \mathbf{I}$. Find the optimal solution of the primal problem (G) assuming that the condition of part (b) holds. **Hint:** recast the problem as a problem of finding an orthogonal projection of a certain point to a certain set.

12.8 Consider the convex problem

$$\text{(P)} \quad \begin{array}{ll} \min_{\mathbf{x}\in\mathbb{R}^n} & \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1 + \|\mathbf{x}\|_\infty - \sum_{i=1}^n \log(x_i) \\ \text{s.t.} & \mathbf{C}\mathbf{x} \le \mathbf{d}, \\ & \mathbf{x} \ge \mathbf{e}, \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{C} \in \mathbb{R}^{p\times n}, \mathbf{d} \in \mathbb{R}^p$ and $\mathbf{e}$ is the vector of all ones. Find a dual problem of (P). Do not make any transformations that will ruin the convexity of the problem.

12.9 Let $\mathbf{u} \in \mathbb{R}^n_{++}$. Consider the problem

$$\min\left\{\sum_{j=1}^n x_j^2 : \sum_{j=1}^n x_j = 1, 0 \le x_j \le u_j\right\}$$

(a) Write a necessary and sufficient condition (in terms of the vector $\mathbf{u}$) under which the problem is feasible.

(b) Write a dual problem in one variable.

(c) Describe an algorithm for solving the optimization problem using the dual problem obtained in part (b).