

CONTENTS

Preface	xiii
1 Real Analysis and Theory of Functions: A Quick Review	1
Introduction	1
1.1 Sets	2
1.2 Mappings	3
1.3 The axiom of choice and Zorn's lemma	5
1.4 Construction of the sets \mathbb{R} and \mathbb{C}	8
1.5 Cardinal numbers; finite and infinite sets	9
1.6 Topological spaces	11
1.7 Continuity in topological spaces	14
1.8 Compactness in topological spaces	15
1.9 Connectedness and simple-connectedness in topological spaces	16
1.10 Metric spaces	18
1.11 Continuity and uniform continuity in metric spaces	21
1.12 Complete metric spaces	22
1.13 Compactness in metric spaces	23
1.14 The Lebesgue measure in \mathbb{R}^n ; measurable functions	25
1.15 The Lebesgue integral in \mathbb{R}^n ; the basic theorems	28
1.16 Change of variable in Lebesgue integrals in \mathbb{R}^n	33
1.17 Volumes, areas, and lengths in \mathbb{R}^n	34
1.18 The spaces $C^m(\Omega)$ and $C^m(\bar{\Omega})$; domains in \mathbb{R}^n	36
2 Normed Vector Spaces	43
Introduction	43
2.1 Vector spaces; Hamel bases; dimension of a vector space	44
2.2 Normed vector spaces; first properties and examples; quotient spaces	47
2.3 The space $\mathcal{C}(K; Y)$ with K compact; uniform convergence and local uniform convergence	53
2.4 The spaces ℓ^p , $1 \leq p \leq \infty$	57
2.5 The Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$	61
2.6 Regularization and approximation in the spaces $L^p(\Omega)$, $1 \leq p < \infty$	68
2.7 Compactness and finite-dimensional normed vector spaces; F. Riesz theorem	76
2.8 Application of compactness in finite-dimensional normed vector spaces: The fundamental theorem of algebra	79

2.9	Continuous linear operators in normed vector spaces; the spaces $\mathcal{L}(X; Y)$, $\mathcal{L}(X)$, and X'	82
2.10	Compact linear operators in normed vector spaces	89
2.11	Continuous multilinear mappings in normed vector spaces; the space $\mathcal{L}_k(X_1, X_2, \dots, X_k; Y)$	91
2.12	Korovkin's theorem	97
2.13	Application of Korovkin's theorem to polynomial approximation; Bohman's, Bernstein's, and Weierstraß' theorems	100
2.14	Application of Korovkin's theorem to trigonometric polynomial approximation; Fejér's theorem	104
2.15	The Stone–Weierstraß theorem	109
2.16	Convex sets	114
2.17	Convex functions	118
3	Banach Spaces	123
	Introduction	123
3.1	Banach spaces; first properties	124
3.2	First examples of Banach spaces; the spaces $\mathcal{C}(K; Y)$ with K compact and Y complete, and $\mathcal{L}(X; Y)$ with Y complete	130
3.3	Integral of a continuous function of a real variable with values in a Banach space	133
3.4	Further examples of Banach spaces: the spaces ℓ^p and $L^p(\Omega)$, $1 \leq p \leq \infty$	135
3.5	Dual of a normed vector space; first examples; F. Riesz representation theorem in $L^p(\Omega)$, $1 \leq p < \infty$	138
3.6	Series in Banach spaces	148
3.7	Banach fixed point theorem	152
3.8	Application of Banach fixed point theorem: Existence of solutions to nonlinear ordinary differential equations; Cauchy–Lipschitz theorem; the pendulum equation	156
3.9	Application of Banach fixed point theorem: Existence of solutions to nonlinear two-point boundary value problems	161
3.10	Ascoli–Arzelà's theorem	164
3.11	Application of Ascoli–Arzelà's theorem: Existence of solutions to nonlinear ordinary differential equations; Cauchy–Peano theorem; Euler's method	169
4	Inner-Product Spaces and Hilbert Spaces	173
	Introduction	173
4.1	Inner-product spaces and Hilbert spaces; first properties; Cauchy–Schwarz–Bunyakovskiĭ inequality; parallelogram law	174
4.2	First examples of inner-product spaces and Hilbert spaces; the spaces ℓ^2 and $L^2(\Omega)$	181
4.3	The projection theorem	183
4.4	Application of the projection theorem: Least-squares solution of a linear system	193
4.5	Orthogonality; direct sum theorem	195

4.6	F. Riesz representation theorem in a Hilbert space	197
4.7	First applications of the F. Riesz representation theorem: Hahn–Banach theorem in a Hilbert space; adjoint operators; reproducing kernels	199
4.8	Maximal orthonormal families in an inner-product space	205
4.9	Hilbert bases and Fourier series in a Hilbert space	213
4.10	Eigenvalues and eigenvectors of self-adjoint operators in inner-product spaces	219
4.11	The spectral theorem for compact self-adjoint operators	221
5	The “Great Theorems” of Linear Functional Analysis	231
	Introduction	231
5.1	Baire’s theorem; a first application: Noncompleteness of the space of all polynomials	232
5.2	Application of Baire’s theorem: Existence of nowhere differentiable continuous functions	236
5.3	Banach–Steinhaus theorem, <i>alias</i> the uniform boundedness principle; application to numerical quadrature formulas	238
5.4	Application of the Banach–Steinhaus theorem: Divergence of Lagrange interpolation	245
5.5	Application of the Banach–Steinhaus theorem: Divergence of Fourier series	252
5.6	Banach open mapping theorem; a first application: Well-posedness of two-point boundary value problems	255
5.7	Banach closed graph theorem; a first application: Hellinger–Toeplitz theorem	259
5.8	The Hahn–Banach theorem in a vector space	261
5.9	The Hahn–Banach theorem in a normed vector space; first consequences	264
5.10	Geometric forms of the Hahn–Banach theorem; separation of convex sets	272
5.11	Dual operators; Banach closed range theorem	277
5.12	Weak convergence and weak * convergence	286
5.13	Banach–Saks–Mazur theorem	294
5.14	Reflexive spaces; the Banach–Eberlein–Šmulian theorem	297
6	Linear Partial Differential Equations	305
	Introduction	305
6.1	Quadratic minimization problems; variational equations and variational inequalities	306
6.2	The Lax–Milgram lemma	310
6.3	Weak partial derivatives in $L^1_{\text{loc}}(\Omega)$; a brief incursion into distribution theory	312
6.4	Hypoellipticity of Δ	319
6.5	The Sobolev spaces $W^{m,p}(\Omega)$ and $H^m(\Omega)$: First properties	326
6.6	The Sobolev spaces $W^{m,p}(\Omega)$ and $H^m(\Omega)$ with Ω a domain; imbedding theorems, traces, Green’s formulas	331
6.7	Examples of second-order linear elliptic boundary value problems; the membrane problem	338
6.8	Examples of fourth-order linear boundary value problems; the biharmonic and plate problems	355

6.9	Examples of nonlinear boundary value problems associated with variational inequalities; obstacle problems	363
6.10	Eigenvalue problems for second-order elliptic operators	369
6.11	The spaces $W^{-m,q}(\Omega)$ and $H^{-m}(\Omega)$; J.L. Lions lemma	377
6.12	The Babuška–Brezzi inf-sup theorem; application to constrained quadratic minimization problems	382
6.13	Application of the Babuška–Brezzi inf-sup theorem: Primal, mixed, and dual formulations of variational problems	388
6.14	Application of the Babuška–Brezzi inf-sup theorem and of J.L. Lions lemma: The Stokes equations	394
6.15	A second application of J.L. Lions lemma: Korn’s inequality	403
6.16	Application of Korn’s inequality: The equations of three-dimensional linearized elasticity	412
6.17	The classical Poincaré lemma and its weak version as an application of J.L. Lions lemma and of the hypoellipticity of Δ	419
6.18	Application of Poincaré’s lemma: The classical and weak Saint-Venant lemmas; the Cesàro–Volterra path integral formula	429
6.19	Another application of J.L. Lions lemma: The Donati lemmas	437
6.20	Pfaff systems	444
7	Differential Calculus in Normed Vector Spaces	451
	Introduction	451
7.1	The Fréchet derivative; the chain rule; the Piola identity; application to extrema of real-valued functions	452
7.2	The mean value theorem in a normed vector space; first applications	465
7.3	Application of the mean value theorem: Differentiability of the limit of a sequence of differentiable functions	469
7.4	Application of the mean value theorem: Differentiability of a function defined by an integral	472
7.5	Application of the mean value theorem: Sard’s theorem	474
7.6	A mean value theorem for functions of class \mathcal{C}^1 with values in a Banach space	477
7.7	Newton’s method for solving nonlinear equations; the Newton–Kantorovich theorem in a Banach space	478
7.8	Higher order derivatives; Schwarz lemma	500
7.9	Taylor formulas; application to extrema of real-valued functions	507
7.10	Application: Maximum principle for second-order linear elliptic operators	513
7.11	Application: Lagrange interpolation in \mathbb{R}^n and multipoint Taylor formulas	522
7.12	Convex functions and differentiability; application to extrema of real-valued functions	540
7.13	The implicit function theorem; first application: Class \mathcal{C}^∞ of the mapping $A \rightarrow A^{-1}$	548
7.14	The local inversion theorem; the invariance of domain theorem for mappings of class \mathcal{C}^1 in Banach spaces; class \mathcal{C}^∞ of the mapping $A \rightarrow A^{1/2}$	554
7.15	Constrained extrema of real-valued functions; Lagrange multipliers	560
7.16	Lagrangians and saddle-points; primal and dual problems	565

8	Differential Geometry in \mathbb{R}^n	575
	Introduction	575
8.1	Curvilinear coordinates in an open subset of \mathbb{R}^n	576
8.2	Metric tensor; volumes and lengths in curvilinear coordinates	578
8.3	Covariant derivative of a vector field	583
8.4	Tensors — a brief introduction	588
8.5	Necessary conditions satisfied by the metric tensor; the Riemann curvature tensor	595
8.6	Existence of an immersion on an open subset of \mathbb{R}^n with a prescribed metric tensor; the fundamental theorem of Riemannian geometry	598
8.7	Uniqueness up to isometries of immersions with the same metric tensor; the rigidity theorem for an open subset of \mathbb{R}^n	608
8.8	Curvilinear coordinates on a surface in \mathbb{R}^3	613
8.9	First fundamental form of a surface; areas, lengths, and angles on a surface	614
8.10	Isometric, equiareal, and conformal surfaces	622
8.11	Second fundamental form of a surface; curvature on a surface	624
8.12	Principal curvatures; Gaussian curvature	629
8.13	Covariant derivatives of a vector field defined on a surface; the Gauß and Weingarten formulas	636
8.14	Necessary conditions satisfied by the first and second fundamental forms: The Gauß and Codazzi–Mainardi equations	640
8.15	Gauß Theorema Egregium; application to cartography	643
8.16	Existence of a surface with prescribed first and second fundamental forms; the fundamental theorem of surface theory	646
8.17	Uniqueness of surfaces with the same fundamental forms; the rigidity theorem for surfaces	654
9	The “Great Theorems” of Nonlinear Functional Analysis	657
	Introduction	657
9.1	Nonlinear partial differential equations as the Euler–Lagrange equations associated with the minimization of a functional	658
9.2	Convex functions and sequentially lower semicontinuous functions with values in $\mathbb{R} \cup \{\infty\}$	664
9.3	Existence of minimizers for coercive and sequentially weakly lower semicontinuous functionals	671
9.4	Application to the von Kármán equations	674
9.5	Existence of minimizers in $W^{1,p}(\Omega)$	683
9.6	Application to the p -Laplace operator	691
9.7	Polyconvexity; compensated compactness; John Ball’s existence theorem in nonlinear elasticity	693
9.8	Ekeland’s variational principle; existence of minimizers for functionals that satisfy the Palais–Smale condition	711
9.9	Brouwer’s fixed point theorem — a first proof	718
9.10	Application of Brouwer’s theorem to the von Kármán equations, by means of the Galerkin method	726

9.11 Application of Brouwer's theorem to the Navier–Stokes equations, by means of the Galerkin method	728
9.12 Schauder's fixed point theorem; Schäfer's fixed point theorem; Leray–Schauder fixed point theorem	734
9.13 Monotone operators	739
9.14 The Minty–Browder theorem for monotone operators; application to the p -Laplace operator	742
9.15 The Brouwer topological degree in \mathbb{R}^n : Definition and properties	748
9.16 Brouwer's fixed point theorem — a second proof — and the hairy ball theorem	764
9.17 Borsuk's and Borsuk–Ulam theorems; Brouwer's invariance of domain theorem	767
Bibliographical Notes	777
Bibliography	781
Main Notations	807
Index	815