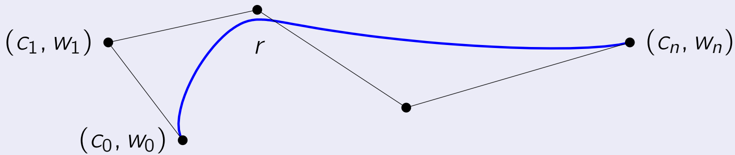
Control Polygon with Weights

A rational Bézier curve r of degree $\leq n$ in $\mathbb{R}^d$ has a rational parametrization in terms of Bernstein polynomials:

$$r(t) = \frac{\sum_{k=0}^{n} (c_k w_k)\, b_k^n(t)}{\sum_{k=0}^{n} w_k\, b_k^n(t)}, \quad 0 \leq t \leq 1,$$

with positive weights w_k and control points $c_k = (c_{k,1}, \ldots, c_{k,d})$.



As for polynomial Bézier curves, the control polygon c qualitatively models the shape of r. The weights give additional design flexibility by controlling the significance of the associated control points.

Weight Points

Scaling the weights, $w_k \to \lambda w_k$, does not change the parametrization of a rational Bézier curve. This extraneous degree of freedom can be eliminated by specifying merely the ratios $w_k : w_{k-1}$. As is illustrated in the figure, these ratios can be visualized as so-called weight points

$$d_k = \frac{w_{k-1}}{w_{k-1} + w_k} c_{k-1} + \frac{w_k}{w_{k-1} + w_k} c_k, \quad k = 1, \ldots, n .$$

The position of d_k within the edge $[c_{k-1}, c_k]$ uniquely determines $w_k : w_{k-1} \in (0, \infty)$ and this eliminates the inherent redundancy of the weights in an elegant fashion.

Affine Invariance

The parametrization

$$[0,1] \ni t \mapsto r(t) = \sum_{k=0}^{n} c_k \, \beta_k^n(t), \quad \beta_k^n = w_k b_k^n \Big/ \sum_{\ell=0}^{n} w_\ell \, b_\ell^n \, ,$$

of a rational Bézier curve is affine invariant, i.e., if we apply an affine transformation

$$x \mapsto Ax + a$$

to r, we obtain the same result as with a transformation of the control points:

$$Ar + a = \sum_{k=0}^{n} (Ac_k + a) \, \beta_k^n \, .$$