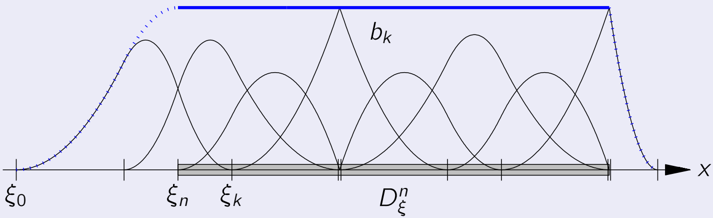
Parameter Interval and Regularity of Knot Sequences

The parameter interval D_ξ^n is the maximal interval on which the B-splines $b_{k,\xi}^n$, $k \sim \xi$, which correspond to the knot sequence ξ, form a partition of unity.



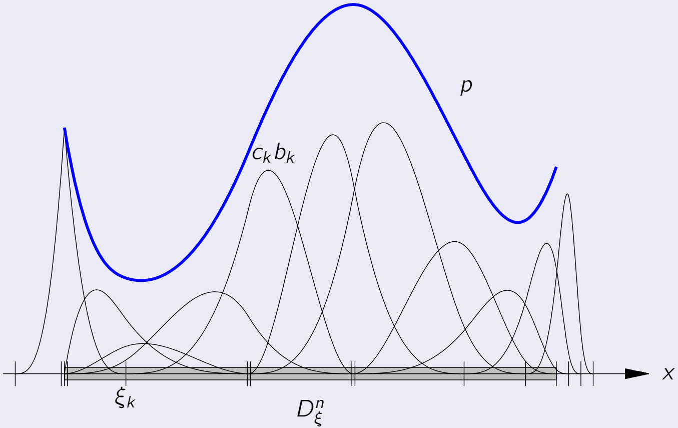
For a bi-infinite knot sequence $D_\xi^n = \mathbb{R}$ and for a finite knot sequence $\xi_0, \ldots, \xi_{m+n}$, $D_\xi^n = [\xi_n, \xi_m]$ unless $\xi_m = \xi_{m+n}$, in which case $b_{m-1,\xi}^n$ is discontinuous at ξ_m and $D_\xi^n = [\xi_n, \xi_m)$.

We say that a knot sequence ξ is n-regular if each B-spline b_k, $k \sim \xi$, is continuous and nonzero at some points in D_ξ^n. More explicitly, n-regularity requires that all knot multiplicities are $\leq n$ and, for a finite knot sequence $\xi : \xi_0, \ldots, \xi_{m+n}$ in addition, that $\xi_n < \xi_{n+1},\ \xi_{m-1} < \xi_m$.

Splines

A spline p of degree $\leq n$ with $n > 0$ is a linear combination of the B-splines corresponding to an n-regular knot sequence ξ:

$$p(x) = \sum_k c_k b^n_{k,\xi}(x), \quad x \in D^n_\xi \, .$$



The coefficients are unique, i.e., the B-splines b_k , $k \sim \xi$, restricted to D_ξ^n form a basis for the spline space denoted by S_ξ^n.

Equivalently, S_ξ^n consists of all continuous functions on the parameter interval D_ξ^n which are

- polynomials of degree $\leq n$ on the nondegenerate knot intervals of D_ξ^n;
- $n - \mu$ times continuously differentiable at an interior knot of D_ξ^n with multiplicity μ.

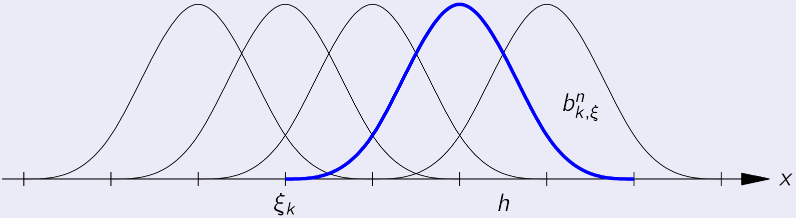
Uniform B-Splines

The standard uniform B-spline b^n has the knots $0, 1, \ldots, n+1$. In this special case, the recursions for evaluation and differentiation simplify:

$$\begin{aligned} nb^n(x) &= xb^{n-1}(x) + (n+1-x)b^{n-1}(x-1), \\ \frac{d}{dx}b^n(x) &= b^{n-1}(x) - b^{n-1}(x-1)\,. \end{aligned}$$

Moreover, the second identity can be written as an averaging process:

$$b^n(x) = \int_0^1 b^{n-1}(x-y)\,dy\,.$$



The uniform B-splines for an arbitrary uniform knot sequence with grid width h, i.e., with knot intervals $\xi_\ell + [0, h]$, are scaled translates of b^n:

$$b^n_{k,\xi}(x) = b^n((x - \xi_k)/h), \quad k \sim \xi \,.$$