Approximation and Modeling with B-Splines

Problem Collection

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Contents

Part I

Problems

1 Polynomials

1.1 Monomial Form

Problem 1.1.1 Nested Multiplication for Hypergeometric Sums
Describe an algorithm for computing

$$\sum_{k=0}^{n} \frac{(a)_k (b)_k}{(1)_k (c)_k} x^k$$

with $(t)_\ell = t(t+1)\cdots(t+\ell-1)$ the Pochhammer symbol. How many operations are needed?
Answer
operations: ☐ n

Problem 1.1.2 Polynomial Division via Nested Multiplication
Show that nested multiplication can be used to divide a polynomial $p_n x^n + \cdots + p_0$ by a linear factor $(x-a)$. The coefficients of the quotient with remainder,

$$q_n x^{n-1} + \cdots + q_1 + \frac{q_0}{x-a},$$

are generated via the recursion

$$q_k = p_k + q_{k+1} a, \quad k = n-1, \ldots, 0,$$

starting with $q_n = p_n$.
Answer
q_0 for $p(x) = x^4 - 5x^3 + 6x - 1$ and $a = 2$: ☐

Problem 1.1.3 Program: Differentiation of a Continued Fraction Recursion
The recursion

$$r_1 = a_1, \quad r_k = x + a_k / r_{k-1},\ k = 2, \ldots, n,$$

defines a rational function r. Write a program `[r,dr] = continued_fraction(a,x)` which computes $r(x)$ and $r'(x)$.
Answer
$\lim_{n\to\infty} r(x)$ for $a_k = 1$ and $x = 2$: ☐

1.2 Taylor Approximation

Problem 1.2.1 Error of the Taylor Polynomial for the Logarithm
Determine an upper bound for the error of the Taylor polynomial of degree $\le n$ at $x_0 = 10$ for the natural logarithm on the interval $[10, 11]$.
Answer
bound for $n = 4$: ☐

Problem 1.2.2 Table of the Exponential via Taylor Approximation
How small should h be chosen in order to approximate $\exp(x)$, $0 \le x \le 1$, from tabulated values at the points $h/2, 3h/2, 5h/2, \ldots, 1 - h/2$ by quadratic Taylor polynomials with error $\le 10^{-12}$?
Answer
$h \le$ ☐ (based on the estimate for the Taylor remainder)

Problem 1.2.3 Cubic Taylor Approximation of a Differential Equation
Approximate the solution $y(x)$ of the initial value problem

$$y' = 3x + 2/y, \quad y(0) = 1,$$

at $x = 1/10$ with the aid of a cubic Taylor polynomial.
Answer
$y(0.1) \approx$ ☐

1.3 Interpolation

Problem 1.3.1 Quadratic Interpolation of Inaccurate Data
Estimate $f(0)$ by interpolating the data

x	1	2	3
$f(x)$	-1	0	4

with a quadratic polynomial p. In the worst case, by how much can $p(0)$ differ if an inaccurate value of $f(3)$ with relative error $\leq 5\%$ is interpolated?
Answer
maximal difference: ☐

Problem 1.3.2 Numerical Differentiation via Quadratic Interpolation
Derive an approximation

$$f'(0) \approx w_0 f(0) + w_1 f(h) + w_2 f(2h)$$

via quadratic interpolation.
Answer
$f'(0)$ for $h = 1/2$ and $f(x) = \sin(\pi x)$: ☐

Problem 1.3.3 4-Point Scheme
If the 4-point scheme is applied to nonperiodic data, for the first and last midpoints different formulas are needed. Determine the weights for the appropriate approximation

$$f_{1/2} \approx \alpha f_0 + \beta f_1 + \gamma f_2 + \delta f_3 .$$

Answer
maximum of the weights $\alpha, \beta, \gamma, \delta$: ☐

Problem 1.3.4 Program: Monomial Form of an Interpolating Polynomial
Write a program `c = aitken_neville(x,f)` which generates the coefficients of an interpolating polynomial via the Aitken–Neville scheme.
Answer
smallest coefficient of the interpolating polynomial to $(k, \exp(k))$, $k = 0, \ldots, 4$: ☐

Problem 1.3.5 Program: Error of Uniform and Tschebyscheff Interpolation for Runge's Function
Write a program `e = runge(n)` which compares the errors of the polynomial interpolants of degree n at uniform (e_1) and Tschebyscheff (e_2) points in $[-1, 1]$ for the function $f(x) = 1/(1 + 25x^2)$.
Answer
maximum of the errors for $n = 8$: ☐

1.4 Bernstein Polynomials

Problem 1.4.1 Conversion of a Quartic Polynomial to Bernstein Form
Write the polynomial

$$x \mapsto 3x^2 - 2x^3 - x^4$$

as linear combination of quartic Bernstein polynomials.
Answer
sum of the Bernstein coefficients: ☐

Problem 1.4.2 Scalar Product of Bernstein Polynomials
Derive a formula for the scalar product

$$\int_0^1 b_k^n(x) b_j^m(x)\, dx$$

of two Bernstein polynomials.
Answer
$\int_0^1 b_3^5 b_2^4$: ☐

1.5 Properties of Bernstein Polynomials

Problem 1.5.1 Integrals of Bernstein Polynomials
Compute

$$\text{a)} \int_0^1 4b_1^3(t) - 5b_2^3(t)\, dt\,, \qquad \text{b)} \int_0^x b_2^3(t)\, dt\,.$$

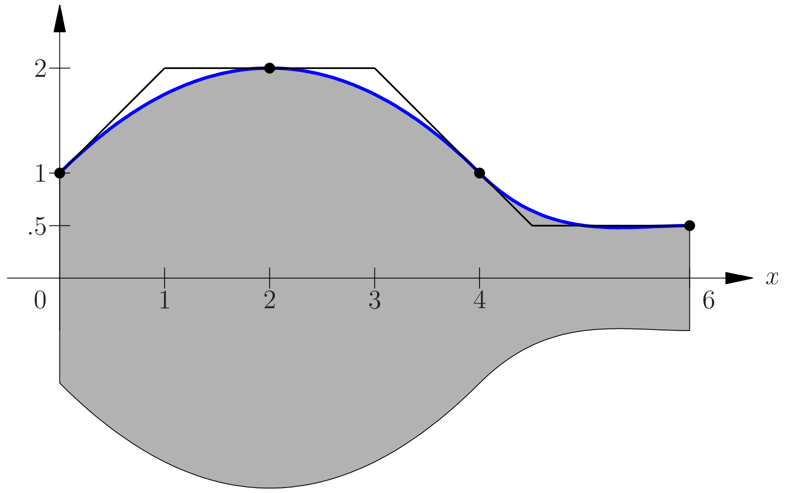
Answer
a) ☐ b) value for $x = 1$: ☐

Problem 1.5.2 Positivity of a Polynomial in Bernstein Form
Show by example that the positivity of a polynomial on the interval $[0, 1]$ does not imply the positivity of its Bernstein coefficients.

1.6 Hermite Interpolant

Problem 1.6.1 Hermite Interpolation of the Cosine
Interpolate the values and derivatives of $\cos(\pi x)$ at the points $x = 0, 1$ by a cubic polynomial p in Bernstein form. Compute the error of the derivatives $p^{(k)}(1/2)$, $k = 0, 1, 2, 3$.
Answer
error of $p'(1/2)$: ☐

Problem 1.6.2 Hermite Representation and Volume of a Wine Bottle
Represent the contour of the wine bottle, depicted in the figure, as a Hermite spline and compute its volume.



Answer
volume: ☐

Problem 1.6.3 Approximation of a Square Root from Hermite Data
Determine an approximation to $\sqrt{x}$ for $x = 2$ from Hermite data at $x = 1, 4$.
Answer
approximation for $\sqrt{2}$: ☐

1.7 Approximation of Continuous Functions

Problem 1.7.1 Accuracy of the Bernstein Approximation
Determine the accuracy of the Bernstein approximation

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} f(k/n) b_k^n(x)$$

numerically. To this end write a program `[ea,eb,ec] = accuracy_bernstein(degrees)` which, for the functions

$$\text{a) } f(x) = |2x - 1|, \quad \text{b) } f(x) = \sqrt{x}, \quad \text{c) } f(x) = \exp(x)\,,$$

computes and plots the errors $e_n = \max_{x \in [0,1]} |f(x) - p_n(x)|$ for a sequence of degrees n.
Answer
smallest of the errors for $n = 1, \ldots, 8$, computed at $x = 0, 0.01, 0.02, \ldots, 1$: ☐

Problem 1.7.2 Convergence of Derivatives of the Bernstein Approximation
Show that, for a smooth function f, the derivatives of the Bernstein approximations

$$p_n = \sum_{k=0}^{n} f(k/n)\, b_k^n$$

converge to f'.

2 Bézier Curves

2.1 Control Polygon

Problem 2.1.1 Vertex and Symmetry Axis of a Parabola
Determine the vertex and the symmetry axis of the parabola parametrized by

$$p = (1,0)\, b_1^2 \, + \, (0,2)\, b_2^2 \, .$$

Answer
axis of symmetry: $\parallel$ ([ ], 1)

Problem 2.1.2 Nonexistence of Polynomial Parametrizations for a Circle
Show that a circular arc cannot be represented as a (polynomial) Bézier curve.

2.2 Properties of Bézier Curves

Problem 2.2.1 Tangent for a Bézier Curve with Multiple Control Points
Show that, for a Bézier curve with a multiple control point

$$c_0 = c_1 = \cdots = c_{k-1} \neq c_k ,$$

the tangent at the left endpoint is parallel to $c_k - c_0$. Does the curve, in general, possess a smooth regular parametrization?
Answer
number of continuous derivatives of a regular parametrization q ($|q'(0)| \neq 0$): [ ]

Problem 2.2.2 Cubic Bézier Approximation of a Semi-Circle
Determine the control points of a cubic Bézier curve p which touches a semi-circle at the points

$$(-1,0)\, , \, (0,1)\, , \, (1,0)\, ,$$

and compute the maximal deviation of $|p(t)|$ from the radius 1.
Answer
$\max_{0 \le t \le 1} |\, |p(t)| - 1|$: [ ]

2.3 Algorithm of de Casteljau

Problem 2.3.1 Evaluation of a Cubic Bézier Curve with de Casteljau's Algorithm
Evaluate the cubic Bézier curve with control points

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}^t$$

at $t = 1/3$ with de Casteljau's algorithm.
Answer
smallest coordinate: [ ]

Problem 2.3.2 Program: Spherical Version of de Casteljau's Algorithm
Write a program `p = casteljau_sphere(C,t)` which implements de Casteljau's algorithm for control points on the unit sphere. To this end replace the weights t and $1-t$ by

$$\sin(t\varphi)/\sin(\varphi) \quad \text{and} \quad \sin((1-t)\varphi)/\sin(\varphi)\,,$$

where φ is the angle between adjecent points in the triangular scheme.
Answer
largest coordinate of the point generated for $C = [1,0,0;\ 0,1,0;\ 0,0,1;\ 1,0,0]$ and $t = 1/3$: ☐

2.4 Differentiation

Problem 2.4.1 Approximation of the Exponential Function by a Quadratic Bézier Curve
Approximate the graph of the exponential function $C:\ y = \exp(x)$ by a regular quadratic Bézier curve $t \mapsto (x(t), y(t))$ near $x = 0$, i.e., determine the Taylor coefficients of x and y so that

$$y(t) - \exp(x(t)) = O(t^m)$$

with m as large as possible. Compare with the accuracy of the quadratic Taylor polynomial $x \mapsto y = 1 + x + x^2/2$.
Answer
maximal order m: ☐

Problem 2.4.2 Program: Distance from a Bézier Curve
Write a program `[d,p] = distance_bezier(C,q)` which computes the distance d of a point q from a Bézier curve with control points C. The program also returns the closest point to q on the curve.
Answer
distance of $(1,1)$ from the Bézier curve with control points $(0,1), (0,0), (2,0)$: ☐

2.5 Curvature

Problem 2.5.1 Curvature at the Midpoint of a Cubic Bézier Curve
Compute the curvature of the Bézier curve with control points

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\mathrm{t}}$$

at the midpoint of the parameter interval.
Answer
$\kappa(1/2)$: ☐

Problem 2.5.2 Interpolation of an Ellipse with a Cubic Bézier Curve
Match position and tangent direction of the ellipse $E:\ x_1^2/9 + x_2^2/4 = 1$ by a cubic Bézier curve p at the points $(\pm 3, 0)$, $(0, 2)$ and compute the curvature of p at the endpoints.
Answer
curvature: ☐

Problem 2.5.3 Smooth Extension of a Quadratic Bézier Curve
For the control points

$$C^- = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{\mathrm{t}}$$

determine all control points C^+ for which the corresponding quadratic Bézier curves $p^\pm$ join with second order contact at $(0,0)$.
Answer
$c_1^+ = (2,0) \rightsquigarrow c_2^+ = (?, \boxed{})$

Problem 2.5.4 Bézier Curve with Second Order Contact with a Circle
Determine all regular planar quadratic Bézier curves which have second order contact with the unit circle at their endpoint $p(0) = (1,0)$.
Answer
$c_1 = (?, 2) \rightsquigarrow c_2 = (\boxed{}, ?)$

2.6 Subdivision

Problem 2.6.1 Subdivision, Tangent Direction, and Curvature of a Quadratic Bézier Curve
Split the Bézier curve with control points

$$C = \begin{pmatrix} 1 & 4 & 7 \\ 0 & 6 & 3 \end{pmatrix}^t$$

and parametrization $t \mapsto p(t)$ at $t = 2/3$ and determine the tangent direction and curvature at $p(2/3)$.
Answer
curvature: $\boxed{}$

Problem 2.6.2 Program: Extension of Bézier Curves via an Inversion of de Casteljau's Algorithm
Explain how to invert de Casteljau's subdivision scheme

$$p \mapsto (p_{\mathrm{left}}, p_{\mathrm{right}}),$$

i.e., how to determine p from p_{left}. Write a program `D = bezier_extension(C,s)` which computes the control points d_k of the extension of the Bézier curve with control points c_k to the interval $[0, s]$, $s > 1$.
Answer
largest coordinate of d_k for $s = 4$ and $C = [0\,2; 1\,5; 3\,6]$: $\boxed{}$

Problem 2.6.3 Reduction of the Edge Length of the Control Polygon by de Casteljau's Algorithm
Show that the maximal edge length of the control polygon of a Bézier curve is reduced by at least a factor 1/2 by subdivision at the midpoint.

2.7 Geometric Hermite Interpolation

Problem 2.7.1 Geometric Hermite Interpolation of an Ellipse
Approximate the right half of the ellipse

$$E: \; x_1^2 + x_2^2/4 = 1$$

by a cubic Bézier curve via geometric Hermite interpolation.
Answer
first coordinate of the middle control points c_1 and c_2: $\boxed{}$

3 Rational Bézier Curves

3.1 Control Polygon and Weights

Problem 3.1.1 Interpolation with a Rational Quadratic Bézier Curve
Show that, with an appropriate choice of the weights, a rational quadratic Bézier curve can interpolate any point in the interior of the triangle formed by its control points. Determine, in particular, weights for the curve which passes through the center $(c_0 + c_1 + c_2)/3$ of the control polygon.
Answer
middle weight for a parametrization in standard form:

Problem 3.1.2 Limit for Movement of Weight Points
Determine the limit of a point $r(t)$, $t \in (0,1)$, on a rational Bézier curve as a weight point d_k is moved towards the control point

$$\text{a) } c_{k-1}, \quad \text{b) } c_k$$

and the position of the other weight points is kept fixed.
Consider the example

$$C = \left(\begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 2 & 3 & 2 & 1 \end{array}\right)^{\mathrm{t}}$$

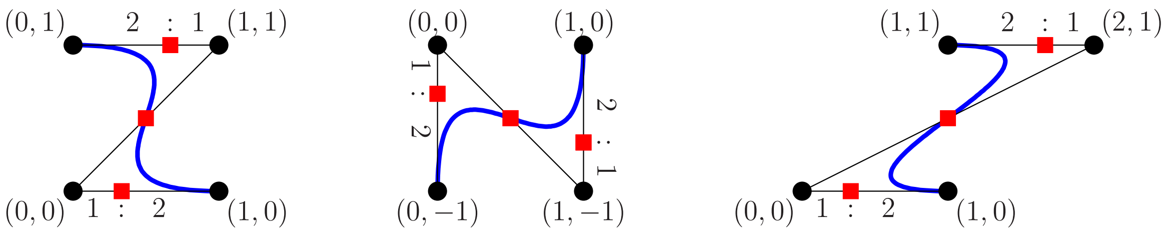
and $k = 3\,,\ t = 2/3\,.$
Answer
largest coordinate of the limit point for the example: a) , b)

Problem 3.1.3 Limit and Curve Type for Movement of Weight Points
Determine the limit of the point $r(1/3)$ for a rational Bézier parametrization r with control points $(1,0)$, $(0,0)$, $(0,1)$ and weight points $(\alpha,0)$, $(0,\alpha)$ as

$$\text{a) } \alpha \to 0 \qquad \text{b) } \alpha \to 1\ .$$

Determine also the type of the corresponding conic section, depending on α.
Answer
largest coordinate of the limit for $\alpha \to 1$:

Problem 3.1.4 Determination of Curve Points via Affine Invariance
Determine the points $r(1/3)$ for the three rational Bézier curves r with control and weight points shown in the figure.



Answer
largest coordinate of the three points:

3.2 Basic Properties

Problem 3.2.1 Rational Cubic Parametrizations of a Semi-Circle
Determine a rational cubic parametrization of a semi-circle with weights in standard form.
Answer
middle weights: ☐

Problem 3.2.2 Limit of a Rational Bézier Curve as Two Weights Approach Infinity
Determine the limit of a point $r(t)$ of a rational Bézier curve as the weights of two inner control points tend to infinity with the same rate ($w_j = w_k = \lambda \to \infty$) and the other weights remain fixed. Consider

$$C = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}^{\mathrm{t}}, \; j = 1 \,, \; k = 2$$

and $t = 1/3$ as a concrete example.
Answer
largest coordinate of the limit point: ☐

Problem 3.2.3 Rational Parametrization of a Polynomial Bézier Curve
Show that a rational Bézier parametrization with weights γ^k, $\gamma \in (0, \infty)$, describes a polynomial curve.

Problem 3.2.4 Rational Quadratic Bézier Curve with an Infinite Control Point
For the parametrization

$$t \mapsto r(t) = \frac{c_0 w_0 b_0^2(t) + d b_1^2(t) + c_2 w_2 b_2^2(t)}{w_0 b_0^2(t) + w_2 b_2^2(t)}, \quad t \in [0,1]\,,$$

d is regarded as an infinite control point. Justify this terminology by considering r as the limit of standard rational Bézier parametrizations. Moreover, compute the tangent vectors at the endpoints.
Answer
largest component of the tangent vectors for $c_0 = (0,0)$, $d = (0,1)$, $c_2 = (1,0)$ and $w_0 = 1$, $w_2 = 2$: ☐

3.3 Algorithms

Problem 3.3.1 Subdivision of a Rational Bézier Curve
Subdivide the rational Bézier curve with

$$(C \,|\, w) = \left(\begin{array}{cc|c} 5 & 7 & 3 \\ 0 & 2 & 9 \\ 8 & 10 & 6 \end{array}\right)$$

at the parameter $t = 1/3$.
Answer
largest coordinate of the common control point: ☐

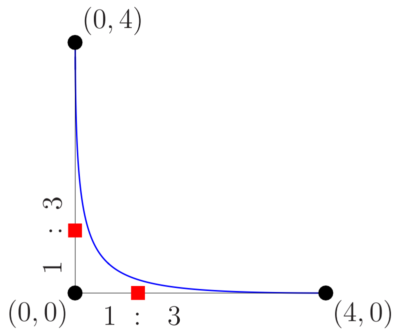
Problem 3.3.2 Program: Offsets of a Rational Bézier Curve
Write a program `[r1,r2] = offset_curve(C,w,d,t)` which evaluates the offset curves with distance d (parallel curves with a prescribed distance) of a planar rational Bézier curve with control points C and weights w at the parameters t, ignoring self-intersections.

Problem 3.3.3 Evaluation and Standard Form of a Rational Bézier Curve
Evaluate the rational quadratic Bézier curve with

$$\left(\, C \mid w \,\right) = \left(\begin{array}{cc|c} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 2 & 2 & 3 \end{array}\right)$$

at the midpoint of the parameter interval via de Casteljau's algorithm. Determine the standard weights and the type (ellipse, parabola, or hyperbola) of the curve.
Answer
largest coordinate of the curve midpoint: ☐

Problem 3.3.4 Weight Points and de Casteljau's Scheme for a Rational Quadratic Bézier Curve
The figure shows a control polygon with weight points for a rational Bézier curve r.



Determine weights for r and compute $r(1/2)$. Is the curve part of an ellipse or of a hyperbola?
Answer
largest coordinate of $r(1/2)$: ☐

3.4 Conic Sections

Problem 3.4.1 Distinction Between an Ellipse and a Hyperbola in Terms of Their Control Points
Show that a rational quadratic Bézier curve, which parametrizes an ellipse (hyperbola), intersects the line segment from the control point c_1 to $(c_0 + c_2)/2$ after (before) the midpoint.

Problem 3.4.2 Parametrization of a Segment of a Hyperbola
Determine a rational quadratic parametrization of the segment of a hyperbola, defined implicitly by $xy = 1$, $0 < x, y \leq a$.
Answer
middle weight of a standard parametrization for $a = 2$: ☐

Problem 3.4.3 Program: Matrix Representation of a Rational Quadratic Bézier Curve
Write a program `A = bezier_quadric(C,w)` which determines the matrix representation

$$v^{\mathrm{t}} A v = 0, \quad v = (p_1, p_2 \mid q)^{\mathrm{t}}$$

of a rational quadratic Bézier parametrization $(p_1/q, p_2/q)$ with control points C and weights w.
Answer
$\max_{j,k} |a_{j,k}| \,/\, \min_{j,k} |a_{j,k}|$ for $C = [1\,0; 0\,0; 0\,1]$ and $w = [1; 1/2; 1]$: ☐

Problem 3.4.4 Implicit Form of a Rational Quadratic Bézier Curve
Determine the implicit form of the conic section corresponding to the rational quadratic Bézier curve with

$$(C \mid w) = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1/2 \\ 0 & 2 & 1 \end{array}\right) .$$

Answer
quotient of the coefficents of x^2 and y^2 in the implicit equation: [ ]

Problem 3.4.5 Parametrizations of a Hyperbola
Determine all rational quadratic parametrizations of the hyperbola $Q: \; x_1 x_2 = 1$.

4 B-Splines

4.1 Recurrence Relation

Problem 4.1.1 Values of a Cubic B-spline
Evaluate the B-spline with the knots $0, 1, 3, 3, 4$ at the integers.
Answer
largest of the computed values: ☐

Problem 4.1.2 Values and Derivatives of Quadratic and Cubic B-splines
Compute the values at the points 1 and 2 for the two quadratic B-splines $b^2_{k,\xi}$ corresponding to the knot sequence $\xi_0 = 0, 1, 2, 2, 4 = \xi_4$. Moreover, determine the values and first derivatives of $b^3_{0,\xi}$ at the same points.
Answer
smallest absolute value of the computed results: ☐

Problem 4.1.3 Values and Derivatives of a Quadratic B-spline
For the B-spline b with knots $0, 2, 5, 5$, compute $b(1)$, $b'(1)$, $b(3)$, $b'(3)$.
Answer
largest of the computed results: ☐

Problem 4.1.4 Polynomial Segments of Quadratic and Quartic B-Splines
Determine the polynomial segments of the B-splines with the knot vectors

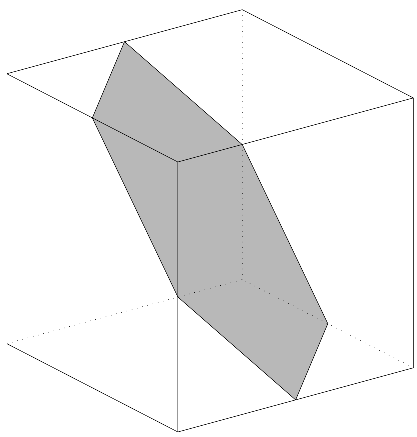
$$\text{a) } (0,0,1,1), \quad \text{b) } (0,1,1,2), \quad \text{c) } (0,0,1,1,2,2) \ .$$

Answer
values of the B-splines at $x = 1/2$: a) ☐, b) ☐, c) ☐

Problem 4.1.5 Program: B-Splines with Support on a Common Knot Interval
Write a program `p = bspline_polynomials(xi)` which determines for all B-splines $b^n_{k,\xi}$ corresponding to a knot sequence $\xi_0, \ldots, \xi_{2n+1}$ the monomial form $p_{k,0} + p_{k,1}x + \cdots + p_{k,n}x^n$ of the polynomial segments on the common knot interval $[\xi_n, \xi_{n+1}]$.
Answer
largest coefficient of the monomials x^k for the knots $0, 1, 3, 6, 10, 15$: ☐

Problem 4.1.6 B-Spline, defined via Cross Sections of a Cube

Denote by $b(t)$ the area of the intersection of the plane $H:\ x+y+z=t$ with the unit cube $[0,1]^3$. Show that b is a multiple of a B-spline.



Answer
quotient of b and the appropriate B-spline: ☐

Problem 4.1.7 Integral of a Uniform B-Spline
Show that $\int_{\mathbb{R}} b^n = 1$.

4.2 Differentiation

Problem 4.2.1 Third Derivative of a Cubic B-Spline
Determine the third derivative of the B-spline with knots $0, 1, 3, 6, 10$.
Answer
largest value of the third derivative: ☐

Problem 4.2.2 Program: Highest Derivatives of a B-spline
Write a program `d = bspline_derivatives(xi)` which computes the n-th order derivatives of a B-spline of degree n.
Answer
sum of the derivatives for $\xi = (1, 2, 2, 3, 3, 3)$: ☐

4.3 Representation of Polynomials

Problem 4.3.1 Spline with Polynomial Coefficients
Which polynomial p does the spline

$$\sum_{k\in\mathbb{Z}} k^2 b^2(x-k)$$

represent?
Answer
sum of the coefficients of the monomials for p: ☐

Problem 4.3.2 Marsden's Identity for Bernstein Polynomials
Specialize Marsden's identity for the interval $[0, 1]$ and $(n + 1)$-fold knots at 0 and 1. Give a direct proof of the resulting formula for Bernstein polynomials.
Answer
largest Marsden coefficient for $(x - 1/2)^3$: ☐

Problem 4.3.3 Program: Representation of Polynomials
Write a program `q = marsden_polynomials(p)` which determines the coefficients $q(k) = q_1 + q_2 k + \cdots$ in the representation of a polynomial p of degree $\le n$ by the B-splines $b^n_{k,\xi}$ with uniform knots $\xi_k = k \in \mathbb{Z}$.
Answer
largest monomial coefficient q_k for $p(x) = 1 + x + \cdots + x^5$: ☐

4.4 Splines

Problem 4.4.1 Basis of Truncated Powers for Simple Knots
Show: For simple knots $\xi_0, \ldots, \xi_{m+n}$, the monomials and the so-called truncated powers,

$$1, x, \ldots, x^n, \quad \varphi_k(x) = (\max(0, x-\xi_k))^n\,,\; k = n+1, \ldots, m-1\,,$$

form a basis for the spline space S^n_ξ.

4.5 Evaluation and Differentiation

Problem 4.5.1 Evaluation of a Cubic Spline
Evaluate the cubic spline p with coefficients $c = (2, -4, 5, 5, 2, -2)$ and knots $\xi = (0, 1, 1, 1, 2, 2, 4, 5, 6, 7)$ at the points $x = 1, 2, 3, 4$.
Answer
$p(1) + \cdots + p(4)$: ☐

Problem 4.5.2 Program: Integrating a Spline
Write a program `[d,eta] = spline_integrate(c,xi)` which determines B-spline coefficients d_k and a knot sequence η of an indefinite integral of a spline with coefficients c_k and knots ξ_ℓ.
Answer
$\max_k d_k - \min_k d_k$ for a cubic spline with $\xi_k = 2^k$ and $c_k = \xi_k$, $k = 0, \ldots, 5$: [ ]

Problem 4.5.3 Values of a Uniform Cubic Spline at the Knots
Derive a formula for the values of a uniform cubic spline at the knots.
Answer
absolute value of the spline at a knot for $c_k = (-1)^k$: [ ]

Problem 4.5.4 Program: B-Spline Coefficients and Knots of a Cubic Spline from Hermite Data
Write a program `[c,xi] = spline_hermite(x,p,d)` which determines the B-spline coefficients c_k and a knot sequence ξ of a cubic spline from the values p_k and the derivatives d_k at the double knots x_k.
Answer
largest coefficient c_k for $x_k = p_k = d_k = k^2$, $k = 0, \ldots, 5$: [ ]

Problem 4.5.5 Integral of a B-spline
Show that

$$\int_{\mathbb{R}} b_{k,\xi}^n = \frac{\xi_{k+n+1} - \xi_k}{n+1} .$$

Problem 4.5.6 Program: Hermite Data from B-Spline Coefficients
Write a program `[p,dp] = hermite_data(c,xi)` which computes the Hermite data $p(\xi_\ell)$, $p'(\xi_\ell)$ from the B-spline coefficients c of a cubic spline $p \in S_\xi^3$ with double knots.
Answer
largest value of $p(\xi_\ell)$ and $p'(\xi_\ell)$ for $\xi_{2k} = \xi_{2k+1} = k^2$ and $c_k = (-1)^k$, $k = 0, \ldots, 5$: [ ]

Problem 4.5.7 Differentiation of a Cubic Spline
Compute the B-spline coefficients of the first and second derivative of the cubic spline with

$$\xi_0 = 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 4,\ 7,\ 8,\ 9,\ 10 = \xi_{10}, \quad c_0 = 0,\ 3,\ 5,\ 9,\ 9,\ -1,\ -6 = c_6\,,$$

on the parameter interval $D_\xi^3 = [1, 7]$.
Answer
smallest B-spline coefficient of the derivatives: [ ]

Problem 4.5.8 Values and Derivatives of a Quadratic Spline
Compute the values and the derivatives of the quadratic spline with knot sequence $\xi_0 = 0,\ 1,\ 1,\ 2,\ 4,\ 4,\ 5 = \xi_6$ and coefficients $c_0 = 7$, $c_1 = 3$, $c_2 = 6$, $c_3 = 8$ at $x = 2$ and $x = 3$.
Answer
largest of the computed values: [ ]

4.6 Periodic Splines

Problem 4.6.1 Periodic Extension by a Quadratic Spline
Determine the 3-periodic quadratic spline p with simple integer knots $\xi_k = k$ with $p(x) = x$ for $x \in [0, 1]$.
Answer
B-spline coefficient c_{100}: [ ]

Problem 4.6.2 Periodic Extension by a Cubic Spline
For which α does the polynomial

$$p(x) = x^3 - \alpha x, \quad x \in [0, 1]\,,$$

have a 3-periodic extension by a cubic spline with uniform integer knots and what are the B-spline coefficients of p?

Answer

α: [ ]

5 Approximation

5.1 Schoenberg's Scheme

Problem 5.1.1 Error of the Derivative for Schoenberg's Scheme
Derive the estimate

$$\max_x |f'(x) - (Qf)'(x)| \le h \max_y |f''(y)|, \quad h = \max_{\xi_k \le x \le \xi_{k+n}} |\xi_{k+n} - \xi_k|,$$

for Schoenberg's scheme of degree n.

Problem 5.1.2 Approximation of Fractional Powers with Schoenberg's Scheme
Construct knot sequences $\xi_0, \ldots, \xi_{m+2}$, for which Schoenberg's scheme with quadratic splines approximates the function x^α for a given exponent $\alpha \in (0,1) \cup (1,2)$ on the interval $[0,1]$ with error less than $c(\alpha)\, m^{-2}$.
Answer
exponent $\beta(\alpha)$ for the ansatz $\xi_\ell = ((\ell-2)/(m-2))^\beta$ and $\alpha = 3/2$: ☐

Problem 5.1.3 Error of Schoenberg's Approximation Applied to the Standard Parabola
Determine the error of Schoenberg's approximation,

$$f \approx Qf = \sum_k f(\xi_k^n)\, b_{k,\xi}^n\,,$$

to the function $f(x) = x^2$ for quadratic splines with uniform knots $\xi_k = kh\,,\ k \in \mathbb{Z}$.
Answer
absolute value of the error: ☐ h^2

Problem 5.1.4 Error of Schoenberg's Scheme for the Exponential Function
Show that the relative error of Schoenberg's approximation with uniform knots kh, $k \in \mathbb{Z}$, of the exponential function is $\ge ch^2$, where c is a positive constant, depending on the degree n.
Answer
largest constant c for $n = 2$: ☐

Problem 5.1.5 Convergence of Schoenberg's Scheme for Continuous Functions
Show that, for a bi-infinite knot sequence ξ and a continuous function f, Schoenberg's approximation,

$$f \approx Qf = \sum_k f(\xi_k^n)\, b_{k,\xi}^n\,,$$

converges for any $x \in \mathbb{R}$, as the maximal length h of the knot intervals tends to 0.

5.2 Quasi-Interpolation

Problem 5.2.1 Quasi-Interpolant for Linear Splines
Construct functionals of the form

$$Q_k f = \alpha_k f(\eta_k) + \beta_k f(\eta_{k+1}), \quad \eta_k = (\xi_k + \xi_{k+1})/2\,,$$

for a quasi-interpolant with linear splines.
Answer
α_k for $\xi_k = 2^k$: ☐

Problem 5.2.2 Functionals for a Quasi-Interpolant for Quadratic Bernstein Polynomials
Construct a quasi-interpolant for quadratic B-splines corresponding to the knot sequence $\xi_0 = -1,\, 0,\, 0,\, 1,\, 1,\, 2 = \xi_5$ with functionals of the form

$$Q_k f = \alpha_k f(0) + \beta_k f(1/2) + \gamma_k f(1)\,.$$

Answer
largest of the coefficients α_k, β_k, and γ_k: ☐

Problem 5.2.3 Quasi-Interpolant for Quadratic Splines
Construct functionals of the form

$$Q_k f = \sum_{\nu=0}^{2} w_{k,\nu} f(\eta_{k+\nu}), \quad \eta_k = (\xi_k + \xi_{k+1})/2\,,$$

for a quasi-interpolant of maximal order with quadratic splines.
Answer
largest weight $w_{0,\nu}$ for $\xi_0 = 0, 1, 4, 9 = \xi_3$: ☐

Problem 5.2.4 Quasi-Interpolant for Uniform Quadratic Splines
Construct functionals of the form

$$Q_k f = \alpha f(k) + \beta f(k + 3/2) + \gamma f(k + 3)$$

for a quasi-interpolant with uniform quadratic B-splines $b^2_{k,\xi}$, $\xi_k = k \in \mathbb{Z}$. Compute $Q_k b^2_{\ell,\xi}$ and explain why $f \mapsto \sum_k (Q_k f)\, b^2_{k,\xi}$ is not a projector.
Answer
α: ☐

Problem 5.2.5 Quasi-Interpolant for Uniform Cubic Splines with Double Knots
Determine quasi-interpolant functionals of the form

$$Q_{2k+\nu} f = \sum_{\mu=0}^{4} w_{\nu,\mu}\, f(kh + \mu h/2)$$

for cubic splines with double knots $\xi_{2k+\nu} = kh$, $k \in \mathbb{Z}$, $\nu \in \{0, 1\}$.
Answer
largest entry of $(w_{2k,0}, \ldots, w_{2k,4})$ for the choice $w_{2k,4} = 0$: ☐

Problem 5.2.6 Program: Quasi-Interpolant for Uniform Splines Based on Values at Knots
Write a program `w = quasi_interpolant(n)` which computes the coefficients w_ν with minimal norm $(\sum_\nu w_\nu^2)^{1/2}$ of functionals

$$Q_k f = \sum_{\nu=0}^{n+1} w_\nu f(\xi_{k+\nu})$$

for a quasi-interpolant of maximal order with splines of degree $\leq n$ with uniform knots $\xi_\ell = \ell h$.
Answer
largest coefficient w_ν for $n = 4$: ☐

5.3 Accuracy of Quasi-Interpolation

Problem 5.3.1 Program: Adaptive Quasi-Interpolation with Quadratic Splines
Write a program `[c,xi] = adaptive(f,tol)` which implements adaptive knot insertion to approximate a function f on $[0,1]$ by a quadratic spline $\sum_k c_k b_{k,\xi}^2$ with estimated error less than a given tolerance tol by quadratic quasi-interpolation. Start with the knot sequence $\xi:\ -0.1, 0, 0, 0.1, \ldots, 0.9, 1, 1, 1.1$, use the standard projector with functionals

$$Q_k f = -f(\xi_{k+1})/2 + 2f(\eta_k) - f(\xi_{k+2})/2, \quad \eta_k = (\xi_{k+1} + \xi_{k+2})/2\,,$$

and add points η_k, where the error is $\geq$ tol, as new knots.
Answer
length of the final knot sequence for the method outlined in the hint, $f(x) = \sqrt{x}$, and tol $= 10^{-6}$:
[ ]

5.4 Stability

No problems for this section.

5.5 Interpolation

Problem 5.5.1 Failure of Diagonal Dominance for Interpolation with Cubic Splines
Show that the matrix

$$b_{k,\xi}^3(t_j)\,, \quad 1 \leq j, k \leq m\,,$$

for cubic spline interpolation at the knots $t_j = \xi_{j+2}$ is, in general, not diagonally dominant.
Answer
not diagonally dominant for $\xi_{j+2} = q^j$ if $q \geq$ [ ]

Problem 5.5.2 Interpolation Matrix for Cubic Splines for the Not-A-Knot Condition
Determine the interpolation matrix for uniform cubic splines with values assigned at the knots, using the not-a-knot boundary condition.
Answer
$\max_\nu |w_\nu| / \min_\nu |w_\nu|$ for the not-a-knot condition $\sum_\nu w_k c_k = 0$: [ ]

Problem 5.5.3 Spline Interpolation of High Order Hermite Data
Show that, for a spline p of degree $\leq n$ with simple knots $\xi_k\,,\ k \in \mathbb{Z}$, the interpolation problem

$$p^{(j)}(\xi_{nk}) = f^{(j)}(\xi_{nk})\,, \quad j = 0, \ldots, n-1\,,\ k \in \mathbb{Z}\,,$$

is uniquely solvable.

Problem 5.5.4 Periodic Cubic Lagrange Spline
Determine the 1-periodic cubic Lagrange spline with the knots

$$\xi_k = k/M\,, \quad k \in \mathbb{Z}\,,$$

which equals 1 at $\xi_0 = 0$, $\xi_{\pm M} = \pm 1$, ...
Answer
B-spline coefficient c_0 for $M = 4$: [ ]

5.6 Smoothing

Problem 5.6.1 Characterization of the Natural Spline Interpolant as Orthogonal Projection

Show that the second derivative p'' of the natural spline interpolant of a function f at the points $x_0, \ldots, x_M$ is the orthogonal projection of f'' with respect to the scalar product

$$\langle \varphi, \psi \rangle = \int_{x_0}^{x_M} \varphi\psi$$

onto piecewise linear functions which vanish at x_0 and x_M.

Problem 5.6.2 Slalom Spline

Show that among all functions f with

$$\alpha_i \le f(x_i) \le \beta_i, \quad i = 0, \ldots, M\,,$$

the natural cubic spline with knots at $x_0 < \cdots < x_M$ minimizes $\int_{x_0}^{x_M} |f''|^2$.

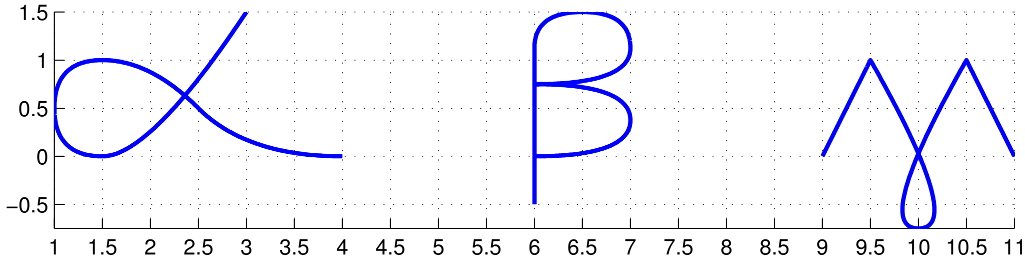
Problem 5.6.3 Limit of the Smoothing Spline as a Weight Tends to Infinity

Prove that the smoothing spline tends to f_k at the knot x_k if the weight w_k tends to infinity.

6 Spline Curves

6.1 Control Polygon

Problem 6.1.1 Approximating Greek Letters with Spline Curves
Construct spline curves which describe, qualitatively correct, the Greek letters α, β, and γ.



Use as few control points as possible.

Problem 6.1.2 Program: Bounding Box Test for Spline Curves
Write a program `x = bounding_box(p,C,n)` which tests with the aid of the bounding boxes if a point p can lie on a spline curve of degree n with control points C.
Answer
result (`true/false`) for $p = [5, 1/2]$, $C = [1, 1;\ 2, -1/2;\ 3, 1/3;\ 4, -1/4;\ \ldots;\ 9, 1/9]$, $n = 2$: ☐

6.2 Basic Properties

Problem 6.2.1 Distance of a Closed Cubic Spline Curve to its Control Polygon
Estimate the distance in the maximum norm of the closed cubic spline curve p with uniform knots and control points

$$C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{\mathrm{t}}$$

to its control polygon.
Answer
bound provided by the standard estimate: ☐

Problem 6.2.2 Distance of a Bézier Curve to its Control Polygon
Specialize the estimate of the distance between a spline curve and its control polygon to the case of Bézier curves.
Answer
$\|p(t) - c(t)\|_\infty \leq$ ☐ $n \max_k \|\Delta^2 c_k\|_\infty$ for even n

Problem 6.2.3 Distance of a Uniform Spline Curve from a Regular m-Sided Control Polygon
Estimate the distance of the closed spline curves of degree $\leq n$ with uniform knots and control points

$$c_k = (\cos(2\pi k/m), \sin(2\pi k/m))$$

with $m \geq 3$ from their control polygons.
Answer
$\|p(t) - c(t)\|_\infty \leq$ ☐ $(n+1)/m^2$

6.3 Refinement

Problem 6.3.1 Refinement of a B-Spline
Express the B-spline with knots 0, 1, 2, 4 as a linear combination of standard uniform B-splines.
Answer
smallest coefficient of the uniform B-splines: [ ]

Problem 6.3.2 Knot Insertion for a Spline Curve
For a cubic spline curve with knots $\tau_k = k$, $k = 0, \ldots, 9$, and control points

$$C = \begin{pmatrix} 0 & 1 & 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}^{\mathrm{t}},$$

insert the knots 3.5 and 6.
Answer
largest coordinate of the newly generated control points: [ ]

Problem 6.3.3 Convergence of Subdivision for Uniform Splines
Show that the control polygons c^0, c^1, ..., generated by the subdivision algorithm for uniform splines, converge to the spline curve:

$$\|c^m(t) - p(t)\|_\infty = O(4^{-m})$$

for all parameters t.

Problem 6.3.4 Knot Removal for a Cubic Spline Curve
Which of the knots $\tau_k = k$, $k = 0, \ldots, 9$, of the cubic spline curve with control points

$$C = \begin{pmatrix} 0 & 4 & 1 & 3 & 6 & 6 \\ 0 & 0 & 3 & 4 & 3 & 0 \end{pmatrix}^{\mathrm{t}}$$

can be removed, and what are the control points corresponding to the coarser knot sequence?
Answer
removable knot: [ ]

Problem 6.3.5 Simultaneous Knot Insertion for Uniform Cubic Splines with Double Knots
Derive a formula for simultaneous knot insertion for cubic spline curves with double knots:

$$\tau : \ldots, 0, 0, h, h, \ldots \quad \rightarrow \quad \tau' : \ldots, 0, 0, h/2, h/2, \ldots$$

Answer
largest weight in the formula $c'_k = \sum_j w_{k,j} c_j$ for the new control points: [ ]

Problem 6.3.6 Knot Insertion for Cubic Splines with Double Knots
Insert the knot 5 twice for the cubic spline curve with

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^{\mathrm{t}}$$

and $\tau_0 = 0,\ 2,\ 2,\ 4,\ 4,\ 6,\ 6,\ 8,\ 8,\ 9 = \tau_9$.
Answer
largest coordinate of the newly generated control points: [ ]

Problem 6.3.7 Knot Insertion for a Quartic Closed Spline Curve
Insert the knot 2 for the quartic closed spline curve with

$$c_0 = (1,0),\ c_1 = (0,1),\ c_2 = (0,0), \quad \tau_0 = 0,\ \tau_1 = 1,\ \tau_2 = 3\,,$$

and periodicity interval $[0,4]$.
Answer
largest coordinate of the newly generated control points: ☐

6.4 Algorithms

Problem 6.4.1 Point on a Quadratic Spline Curve and Tangent Vector
Sketch the quadratic spline curve p with control points

$$C = \begin{pmatrix} 4 & 1 & 1 & 5 \\ 0 & 0 & 3 & 5 \end{pmatrix}^t$$

and knot sequence $1,2,2,3,5,5,6$. Determine $p(4)$ and $p'(4)$ via knot insertion.
Answer
largest coordinate of $p(4)$ and $p'(4)$: ☐

Problem 6.4.2 Point on a Cubic Spline Curve and Tangent Vector
Determine $p(3)$ and $p'(3)$ for the cubic spline curve p with control points

$$C = \begin{pmatrix} -8 & -5 & 5 & 8 & -2 & -6 \\ 2 & -9 & 9 & 6 & -9 & -5 \end{pmatrix}^{\mathrm{t}}$$

and knots 0, 1, 1, 1, 2, 5, 7, 7, 7, 8.
Answer
largest coordinate of $p(3)$ and $p'(3)$: ☐

Problem 6.4.3 Curvature of a Cubic Spline Curve
Derive a formula for the curvature of a uniform cubic spline curve in $\mathbb{R}^3$ at the knots. Give a geometric interpretation. As an example, consider the closed cubic spline curve with the vertices of the unit square $[0,1]^2$ as control points.
Answer
curvature at the knots for the example: ☐

Problem 6.4.4 Bézier Form of a Quadratic Spline Curve
Determine the Bézier form of the closed quadratic spline curve with control points

$$c_0 = (0,0),\ (0,-1),\ (1,0),\ (0,1),\ (-1,0) = c_4$$

and knots $\tau_0 = 0,\ 1,\ 1,\ 3,\ 6 = \tau_4$ in the periodicity interval $[0,10]$.
Answer
largest coordinate of the newly generated control points: ☐

Problem 6.4.5 Program: Variation of a Control Polygon with respect to a Hyperplane
Write a program `v = variation(C,P)` which computes the variation v of a control polygon C with respect to a hyperplane $H \subset \mathbb{R}^d$, determined by d points $P(k,:)$.
Answer
variation for $C = [\cos 0, \sin 0, 0;\ \ldots;\ \cos 10, \sin 10, 10]$, $P = [1,0,0;\ 0,2,0;\ 0,0,3]$: ☐

Problem 6.4.6 Bézier Representation of a Quartic Spline Curve
Determine the Bézier representation of the quartic spline curve with control points

$$C = \begin{pmatrix} 0 & 12 & 12 & 0 & 0 \\ 0 & 0 & 12 & 12 & 0 \end{pmatrix}^{\mathrm{t}}$$

and uniform knots.
Answer
largest coordinate of the Bézier control points: [ ]

6.5 Interpolation

Problem 6.5.1 Program: Interactive Natural Spline Interpolation
Write a program `interactive_interpolation()` which interpolate points in $[0, 10]^2$, specified interactively via mouse click, by a natural cubic spline curve.
Answer
maximal coordinate, rounded to 2 significant digits, of the control point array for the points $[8,\ 2; 2,\ 2; 2,\ 8]$: [ ]

7 Multivariate Splines

7.1 Polynomials

Problem 7.1.1 Representation of a Bilinear Polynomial in Terms of Bernstein Polynomials
Express the polynomial $p(x) = 1 + 2x_1 + 3x_2 + 4x_1x_2$ as a linear combination of bivariate Bernstein polynomials of degree $(2, 2)$.
Answer
largest Bernstein coefficient: [ ]

Problem 7.1.2 Program: Multilinear Interpolation
Write a program `p = multilinear_interpolation(F,x)` which evaluates an n-linear function, defined by the data F at the vertices $\{0, 1\}^n$ of the n-dimensional hypercube $D = [0, 1]^n$ at $x \in D$.
Answer
value for the data $F = [[0,\ 2;\ 1,\ 3], [4,\ 6;\ 5,\ 7]]$ and and $x = [1/2,\ 2/3,\ 3/4]$: [ ]

7.2 Polynomial Approximation

Problem 7.2.1 Zeros of the Error of the Polynomial Orthogonal Projection
Prove that the error $P^n f - f$ of the orthogonal projection onto $\mathbb{P}^n[0, 1]$ has at least $n+1$ simple zeros in $(0, 1)$. Show that this implies

$$\|P^n f - f\|_{\infty,[0,1]} \leq \|f^{(n+1)}\|_{\infty,[0,1]}$$

for any smooth function f.

7.3 Splines

Problem 7.3.1 Program: Values of Multivariate B-Splines
Write a program `b = bspline_multivariate(n)` which generates the d-dimensional array $b^n(k)$, $1 \leq k_\nu \leq n_\nu$, of values of a uniform multivariate B-spline of degree $(n_1, \ldots, n_d)$ at the knots.
Answer
$b^{(4,7,9)}(3, 2, 5) =$ [ ]

Problem 7.3.2 Nonexistence of B-Splines of Minimal Total Degree
Show that, on a bivariate tensor product grid, bivariate B-splines of total degree $\leq n$ and smoothness $n - 1$ do not exist for $n > 1$. More precisely, any function with continuous partial derivatives of order $\leq n-1$, which is a polynomial of total degree $\leq n$ on each grid cell $k+[0, 1]^2$, $k \in \mathbb{Z}^2$, and has support in a square $[0, m]^2$, must vanish identically.

7.4 Algorithms

Problem 7.4.1 Gradient of a Bivariate Biquadratic Spline
Compute the gradient of the spline $p = \sum_k c_k\, b_{k,\xi}^{(2,2)}$ with coefficients

$$\begin{array}{ccccc} & c_{(-2,-2)} = & -8 \quad 4 \quad -8 & & \\ \cdots & & 0 \quad -4 \quad 0 & & \cdots \\ & & 8 \quad 4 \quad -8 & = c_{(0,0)} & \end{array}$$

and uniform knots $\xi_{\nu,\ell} = \ell h$ at $(h/2, h/2)$.
Answer
largest coordinate of the gradient for $h = 1/2$: ☐

Problem 7.4.2 Evaluation of a Bivariate Uniform Spline
Evaluate the uniform bivariate spline $p(x) = \sum_k c_k\, b^{(3,2)}(x - k)$ with coefficients

$$\begin{array}{ccccc} & c_{(0,0)} = & 0 \quad 0 & & \\ \cdots & & 15 \quad 9 & & \cdots \\ \cdots & & 20 \quad 4 & & \cdots \\ & & 30 \quad 18 & = c_{(3,1)} & \end{array}$$

at $x = (3.5, 2)$.
Answer
$p(3.5, 2)$: ☐

7.5 Approximation Methods

Problem 7.5.1 Program: Evaluation of a Multivariate Polynomial
Write a program `P = polynomial_multivariate(C,x)` which evaluates a multivariate polynomial

$$p(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_d=0}^{n_d} c_k x^k$$

at the grid of points $x = (x_{1,j_1}, \ldots, x_{d,j_d})$, $0 \le j_\nu \le n_\nu$.
Answer
largest value of $p(x)$ for $n = (3,3,3)$, $x_\nu = [1:4]$, and $c_k = 1 + k_1 + k_2 + k_3$: ☐

Problem 7.5.2 Program: Multivariate Polynomial Interpolation
Write a program `C = multivariate_interpolation(x,F)` which computes the interpolating polynomial of coordinate degree $(n_1, \ldots, n_d)$ to data $f_{(j_1,\ldots,j_d)}$ on a tensor product grid $(x_{1,j_1}, \ldots, x_{d,j_d})$, $0 \le j_\nu \le n_\nu$.
Answer
largest monomial coefficient for $d = 4$, $n_\nu = 3$, $x_{\nu,j} = j$, and $f_j = (-1)^{j_1+j_2+j_3+j_4}$: ☐

7.6 Hierarchical Bases

Problem 7.6.1 Dimension of a Hierarchical Spline Space

Determine the dimension of the bilinear hierarchical spline space on $D = [4, 36] \times [4, 28]$, determined by the tree

$$\Xi : \ \xi^* \to \eta, \tilde{\eta}, \ \eta \to \zeta$$

where

$$\begin{array}{ll} \xi_1^* : \ 0, 4, \ldots, 40, & \xi_2^* : \ 0, 4, \ldots, 32 \\ \eta_1 : \ 2, 4, \ldots, 20, & \eta_2 : \ 2, 4, \ldots, 20 \\ \tilde{\eta}_1 : \ 24, 26, \ldots, 32, & \tilde{\eta}_2 : , 12, 14, \ldots, 24 \\ \zeta_1 : \ 8, 9, \ldots, 20, & \zeta_2 : \ 8, 9, \ldots, 18\,. \end{array}$$

Answer

dimension: ☐

Problem 7.6.2 Program: Adaptive Approximation of the Square Root with Hierarchical Linear Splines

Write a program `xi = square_root_approximation(tol)` which constructs adaptively a linear hierarchical spline for approximating the function $x \mapsto \sqrt{x}$, $x \in [0, 1]$. Use linear interpolation and subdivide a subinterval at the midpoint if the error is larger than tol. Plot the maximal errors on the subintervals of the resulting hierarchical partition $0 = \xi_1 < \cdots < \xi_m = 1$.

Answer

number of intervals to achieve an error ≤ 0.001: ☐

8 Surfaces and Solids

8.1 Bézier Surfaces

Problem 8.1.1 Approximation of a Sphere by Bi-Quadratic Bézier Patches
Determine a fully symmetric approximation of the sphere $S: x_1^2+x_2^2+x_3^2 = 3$ by 6 bi-quadratic Bézier patches. The patches should touch the sphere at $(\pm 1, \pm 1, \pm 1)$ (endpoint interpolation) as well as at $(\pm\sqrt{3}, 0, 0)$, $(0, \pm\sqrt{3}, 0)$, $(0, 0, \pm\sqrt{3})$.
Answer
distance of the middle Bézier control points to the origin: ☐

Problem 8.1.2 Nonexistence of a Tangent Plane for a Bézier Patch with a Degenerate Boundary
Show by example that a Bézier patch with $c_{0,0} = \cdots = c_{n_1,0}$ does, in general, not have a tangent plane at the multiple boundary control point.

Problem 8.1.3 Conditions for Tangent Plane Continuity
Assume that two regular Bézier parametrizations of degree (n, n) share a common boundary curve:

$$q(1, s) = p(0, s), \quad 0 \le s \le 1 .$$

A sufficient condition for tangent plane continuity is

$$\partial_1 q(1, s) = \alpha \partial_1 p(0, s) + (\beta_0(1-s) + \beta_1 s)\partial_2 p(0, s), \quad 0 \le s \le 1 ,$$

with $\alpha > 0$. Express this condition in terms of the control points.

8.2 Spline Surfaces

Problem 8.2.1 Bicubic Model of a Torus
Model a torus with radii 3 and 1 with uniform bicubic splines with 16 control points. The surface should touch the torus at least at 16 points.

Problem 8.2.2 Program: Spline Model of a Möbius Strip
Model a Möbius strip (a rectangular band, connected at opposite ends, twisted by 180°) by a spline surface.

8.3 Subdivision Surfaces

Problem 8.3.1 Program: 4-Point Scheme for Surfaces
Write a program `Q = four_point_scheme_surface(P)` which implements one step of the tensor product 4-point scheme, which is based on bicubic interpolation, for a rectangular quadrilateral mesh of points $P(j, k, :)$.
Answer
$\sum_{i,j,k} q_{i,j,k}$ for $p_{i,j} = [j, k, j^2 + k^2]$: ☐

Problem 8.3.2 Limit Points for the Scheme of Catmull-Clark
Determine the limits of the irregular vertices for the Catmull-Clark algorithm applied to the wireframe of the standard cube $[-1, 1]^3$.
Answer
distance of the vertex limits to the origin: ☐

8.4 Blending

Problem 8.4.1 Coon's Patch for Quadratic Bézier Boundaries
Blend the boundary values

$$p(x_1, 0) = 0,\ p(1, x_2) = x_2,\ p(x_1, 1) = 1,\ p(0, x_2) = x_2^2, \quad 0 \le x_1, x_2 \le 1\,,$$

with a biquadratic polynomial in Bernstein-Bézier form.
Answer
middle Bézier coefficient $c_{1,1}$: ☐

Problem 8.4.2 Program: Visualization of a Coon's Patch
Write a program `p = coons_patch(ps0,ps1,p0t,p1t,n)` which evaluates the parametrization of a Coon's patch, defined by four boundary curves, at a tensor product grid with coordinates $0, 1/n, \ldots, 1$ and visualizes the surface.
Answer
sum of the coordinates of the patch values for the Coon's patch which interpolates the boundary of the surface (quarter of a torus)

$$S:\ (s,t) \mapsto (\cos(\pi t/2)(2+\cos(\pi s)),\ \sin(\pi t/2)(2+\cos(\pi s)),\ \sin(\pi s)), \quad 0 \le s, t \le 1\,,$$

and the grid, defined by $s = t = (0, \ldots, 10)/10$: ☐

8.5 Solids

Problem 8.5.1 Program: Volume of a Bézier Solid
Write a program `volume = solid_volume(C)` which determines the volume of a Bézier solid with regular parametrization and control points C.
Answer
volume for the solid with control points $c_k = k(1 + |k|^2)$, $k_\nu \le 2$: ☐

9 Finite Elements

9.1 Ritz-Galerkin Approximation

Problem 9.1.1 Ritz-Galerkin Approximation with Sine Functions
Determine the Ritz-Galerkin approximation of the boundary value problem

$$-u'' + u = x, \quad u(0) = u(\pi) = 0\,,$$

for the finite elements $x \mapsto \sin(kx)$, $k = 1, \ldots, n$.
Answer
coefficient of $\sin(10\,x)$: ☐

Problem 9.1.2 Program: Ritz-Galerkin Approximation of a Radially Symmetric Poisson problem
Write a program `residuum = residuum_poisson_radial(n)` which computes and plots the Ritz-Galerkin approximation u_n of the radially symmetric Poisson problem

$$-\frac{1}{r}(ru')' = \exp(r^2), \quad u(1) = 0\,,$$

on the unit disc for the basis functions

$$B_1, \ldots, B_n, \quad B_k(r) = 1 - r^{2k}\,.$$

Moreover, the program calculates the maximum norm of the residuum $e_n(r) = -(1/r)(ru_n')' - \exp(r^2)$.
Answer
maximum norm of the residuum for $n = 5$: ☐

Problem 9.1.3 H^1-Error of Univariate Hat-Functions
Derive the error estimate

$$|u - u_h|_1 \le h|u|_2, \quad |v|_k^2 = \int_0^1 |v^{(k)}|^2\,,$$

for piecewise linear interpolants u_h of a function $u \in H^2(0,1)$.

9.2 Weighted B-Splines

Problem 9.2.1 Ritz-Galerkin Integrals of Bilinear B-Splines
Determine the Ritz-Galerkin integrals

$$g_{k,\ell} = \int \operatorname{grad} b_{k,h}^n \operatorname{grad} b_{\ell,h}^n$$

for bivariate tensor product B-splines of degree $n = (1,1)$.
Answer
$\sum_\ell |g_{k,\ell}|$: ☐

Problem 9.2.2 Program: Rvachev Operations for Weight Functions Represented by `m`-Files
Write a program `rfct(w,operation,w1,w2)`, which implements Rvachev's method for weight functions represented by MATLAB `m`-files `w1, w2`, by generating an `m`-file `w` according to the specified Boolean operation (union, intersection, or complement).

9.3 Isogeometric Elements

Problem 9.3.1 Bijectivity of a Bilinear Isoparametric Transformation of the Unit Square
Show that a bilinear isoparametric transformation of the unit square is bijective if and only if the image is a convex quadrilateral.

9.4 Implementation

Problem 9.4.1 Numerical Integration of Bilinear Splines over a Boundary Cell
For $\Omega:\ 0 < x_1, x_2 < 1,\ x_2 < x_1^2$, and bilinear B-splines b_k with grid width 1, determine weights γ_k such that

$$\int_\Omega p(x)\,dx = \sum_k \gamma_k c_k, \quad p = \sum_k c_k b_k\,.$$

Answer
smallest weight γ_k: $\boxed{}$

Problem 9.4.2 Gauß Parameters for a Bivariate Boundary Cell
Using the univariate formula

$$\int_0^1 f \approx \frac{1}{2} f\left(1/2 - \sqrt{3}/6\right) + \frac{1}{2} f\left(1/2 + \sqrt{3}/6\right)\,,$$

determine weights γ_ℓ and nodes (x_ℓ, y_ℓ) for integration over the boundary cell

$$\Omega : 1 - 2xy > 0\,,\ 0 < x, y < 1\,.$$

Answer
approximation of $\int_\Omega x^y\,dxdy$: $\boxed{}$

9.5 Applications

Problem 9.5.1 Strain Tensor for a Radially Symmetric Displacement
Compute the strain tensor $\varepsilon(u)$ for a radially symmetric displacement $u(x) = \varphi(r)x$, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.
Answer
$\sum_{k,\ell} \varepsilon_{k,\ell}$ for $\varphi(r) = r^2$ and $x = (1,1,1)$: $\boxed{}$

Problem 9.5.2 Program: Elasticity Bilinear Form for Hat-Functions on a Tetrahedron
Write a program `G = P_sigma_epsilon_hat(P,lambda,mu)` which computes the $4 \times 4 \cdot (3 \times 3)$ block matrix

$$\int_{[p_1,p_2,p_3,p_4]} \sigma(B_k e_\alpha) : \varepsilon(B_\ell e_\beta)$$

for the hat functions B_j which correspond to the vertices p_j of a tetrahedron and the unit vectors e_1, e_2, e_3.
Answer
$\sum_{k,\ell,\alpha,\beta} |g_{k,\ell,\alpha,\beta}|$ for the standard simplex with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ and $\lambda = \mu = 1$: $\boxed{}$

Part II

Hints

1 Polynomials

1.1 Monomial Form

Problem 1.1.1 Derive a recursion for the summands.

Problem 1.1.2 Compare coefficients in the identity $p(x) = q(x)(x - a) + q_0$.

Problem 1.1.3 Obtain a recursion for r' by differentiating the expression for r_k.

1.2 Taylor Approximation

Problem 1.2.1 Determine the worst case for each factor of the Taylor remainder.

Problem 1.2.2 Determine first an upper bound $r(h)$ for the Taylor remainder. Then solve the equation $r(h) = 10^{-12}$ for h.

Problem 1.2.3 Differentiate the differential equation with the aid of the chain rule, i.e., by using $(d/dx)f(y) = f'(y)y'$.

1.3 Interpolation

Problem 1.3.1 Use the Lagrange form of the interpolating parabola.

Problem 1.3.2 Differentiate the Lagrange form of the interpolating quadratic polynomial.

Problem 1.3.3 Use the Lagrange form of the cubic interpolant.

Problem 1.3.4 Multiplication of a polynomial by $(x + c)$ corresponds to the operation $a \to [0; a] + c * [a; 0]$ on the coefficients $a(k)$.

Problem 1.3.5 Measure the error based on 1000 equally spaced points in $[-1, 1]$.

1.4 Bernstein Polynomials

Problem 1.4.1 Use the matrix describing the basis transformation.

Problem 1.4.2 Observe that the product of two Bernstein polynomials equals, up to a constant factor, a Bernstein polynomial of higher degree.

1.5 Properties of Bernstein Polynomials

Problem 1.5.1 Use the formula for $(b_k^n)'$.

Problem 1.5.2 It suffices to consider quadratic polynomials.

1.6 Hermite Interpolant

Problem 1.6.1 Use endpoint interpolation and the formula for differentiating a linear combination of Bernstein polynomials.

Problem 1.6.2 Apply the formula for the Bernstein coefficients. Use the symmetry of the first two segments.

Problem 1.6.3 Represent the interpolant in Bernstein form.

1.7 Approximation of Continuous Functions

Problem 1.7.1 Use the program `bernstein` to evaluate the Bernstein polynomials.

Problem 1.7.2 Show first that $p_n' = n \sum_{k=0}^{n-1} (f((k+1)/n) - f(k/n)) b_k^{n-1}$ and apply the mean value theorem.

2 Bézier Curves

2.1 Control Polygon

Problem 2.1.1 Convert to monomial form $a_0 + a_1 t + a_2 t^2$ and substitute $s = t + \alpha$ to obtain the standard parametrization of a parabola.

Problem 2.1.2 Write the components p_ν of the Bézier parametrization in monomial form and consider the equation $p_1^2 + p_2^2 = 1$.

2.2 Properties of Bézier Curves

Problem 2.2.1 Show that the Bézier parametrization has the form $p(t) = c_0 + \gamma(c_k - c_0)t^k + O(t^{k+1})$ and substitute $s = t^k$.

Problem 2.2.2 Use symmetry with respect to the vertical axis and determine the middle control points by testing the parametrization at $t = 1/2$.

2.3 Algorithm of de Casteljau

Problem 2.3.1 Recall that the edges of the control polygon are divided in the ratio $(1-t) : t$.

Problem 2.3.2 Note that $\varphi = \arccos\langle p, q\rangle$ for unit vectors p and q.

2.4 Differentiation

Problem 2.4.1 Setting the derivatives at $t = 0$ of $y(t) - \exp(x(t))$ to zero yields nonlinear equations for the derivatives $x^{(k)}(0)$ and $y^{(k)}(0)$.

Problem 2.4.2 Use the MATLAB functions `polyfit` and `roots`.

2.5 Curvature

Problem 2.5.1 Use subdivision and the formula for the curvature at the endpoints of a Bézier curve.

Problem 2.5.2 Use symmetry and the endpoint interpolation property.

Problem 2.5.3 Use the endpoint interpolation property and the formula for the curvature at the endpoints.

Problem 2.5.4 Use the endpoint interpolation property and the formula for the curvature at the endpoints.

2.6 Subdivision

Problem 2.6.1 Use de Casteljau's algorithm, endpoint interpolation, and the formula for the curvature in terms of the control points.

Problem 2.6.2 View the Bézier curve as the left part of its extension and traverse de Casteljau's triangle from left to right, i.e., determine $p_m^0, \ldots, p_m^{n-m}$ from $p_{m-1}^0, \ldots, p_{m-1}^{n-m+1}$.

Problem 2.6.3 Show first that the maximal edge length of the polygons in de Casteljau's algorithm is nonincreasing.

2.7 Geometric Hermite Interpolation

Problem 2.7.1 Use the parametrization $t \mapsto (\cos t, 2\sin t)$ to compute the curvature of the ellipse and note that symmetry is preserved by the interpolant.

3 Rational Bézier Curves

3.1 Control Polygon and Weights

Problem 3.1.1 Write a point $r(t)$ on a quadratic Bézier curve as repeated convex combination of the control points:

$$r(t) = (1-\alpha)((1-\beta)c_0 + \beta c_2) + \alpha c_1 \,.$$

Problem 3.1.2 Divide numerator and denominator of the rational parametrization by w_0 and write

$$w_\ell = (w_\ell/w_{\ell-1})(w_{\ell-1}/w_{\ell-2}) \ldots (w_1/w_0) \,.$$

Problem 3.1.3 First define weights w_k which correspond to the ratios determined by the weight points.

Problem 3.1.4 Note that the middle and right curve can be obtained from the left curve by a rotation and shear transformation, respectively.

3.2 Basic Properties

Problem 3.2.1 Use endpoint interpolation and that, by symmetry, $c_{1,2} = c_{2,2}$ and $w_1 = w_2$.

Problem 3.2.2 Divide numerator and denominator of the parametrization by λ and note that $w_\ell/\lambda \to 0$ for $j, k \neq \ell$.

Problem 3.2.3 Transform the weights to standard form.

Problem 3.2.4 Apply the quotient rule and use the formulas for the derivatives of polynomial Bézier parametrizations.

3.3 Algorithms

Problem 3.3.1 Apply de Casteljau's algorithm to the homogeneous control points.

Problem 3.3.2 No hint available for this problem.

Problem 3.3.3 Use homogeneous coordinates and note that the type is determined by the weights.

Problem 3.3.4 Note that the weight points d_k divide $[c_{k-1}, c_k]$ in the ratio $w_k : w_{k-1}$. Apply de Casteljau's algorithm in homogeneous coordinates.

3.4 Conic Sections

Problem 3.4.1 Write $r(t) = (1-\alpha)c_1 + \alpha(c_0 + c_2)/2$ and determine α in terms of the weights w_k.

Problem 3.4.2 Use endpoint interpolation to determine the control points c_k and determine the middle weight by testing the parametrization in standard form at $t = 1/2$.

Problem 3.4.3 Test the equation $(p_1(t), p_2(t) \,|\, q(t))A(p_1(t), p_2(t) \,|\, q(t))^{\mathrm{t}} = 0$ at 5 points t to obtain a linear system for the matrix entries $a_{j,k} = a_{k,j}$.

Problem 3.4.4 Compare the coefficients of t^ℓ in the equation $(p_1, p_2 \,|\, q)A(p_1, p_2, q)^{\mathrm{t}} = 0$ where $p_\nu(t)/q(t)$ are the components of the rational parametrization.

Problem 3.4.5 Write the polynomials p_1, p_2, and q of the rational parametrization as products of their linear factors.

4 B-Splines

4.1 Recurrence Relation

Problem 4.1.1 Use the formulas for a B-spline on the first and last knot interval as well as the recurrence relation.

Problem 4.1.2 Use the reccurrence relations or, alternatively, the formula for the first and last polynomial segment of a B-spline.

Problem 4.1.3 Use the formula for the first knot interval as well as the recurrence relation.

Problem 4.1.4 Recall that a B-spline with just two different knots coincides with a Bernstein polynomial and use the formula for a B-spline on the first knot interval as well as the recurrence relation.

Problem 4.1.5 Use the B-spline recursion and note that multiplication with a linear function $a_1 + a_2 x$ changes the coefficients of a polynomial $p_1 + p_2 x + \cdots$ according to $p \to [a_1 * p; 0] + [0; a_2 * p]$.

Problem 4.1.6 Split the intersection into triangles determined by the intersections of H with the edges of the cube.

Problem 4.1.7 Use induction and the definition of the uniform B-spline via averaging.

4.2 Differentiation

Problem 4.2.1 Use the recursion for B-spline derivatives.

Problem 4.2.2 Ignore zero denominators and assign `NaN` as derivative values to empty knot intervals.

4.3 Representation of Polynomials

Problem 4.3.1 Combine the representations of the monomials, obtained from Marsden's identity, in a suitable way.

Problem 4.3.2 Recall that the Bernstein polynomial b_k^n coincides on $[0, 1)$ with the B-spline with an $(n + 1 - k)$-fold knot at 0 and a $(k + 1)$-fold at 1.

Problem 4.3.3 Write $p(x)$ as linear combination of $(x - j)^n$ and $q(k)$ as linear combination of $\psi_k^n(j)$.

4.4 Splines

Problem 4.4.1 Since the number of basis functions match, it suffices to show the linear independence of the monomials and truncated powers.

4.5 Evaluation and Differentiation

Problem 4.5.1 Apply the de Boor algorithm.

Problem 4.5.2 Invert the recursion for the coefficients of the derivative of a spline.

Problem 4.5.3 Use symmetry and that the B-spline values at the knots sum to one.

Problem 4.5.4 Note that for each point x_k only two B-splines are relevant. Express p_k and d_k in terms of the B-spline coefficients via the explicit formulas for the first and last B-spline segments and solve the resulting local linear systems.

Problem 4.5.5 Using the formula for differentiating a spline, derive an identity for $\int_{\xi_k}^x b_{k,\xi}^n$ and then let x tend to infinity.

Problem 4.5.6 Use the formula for differentiating a spline and the de Boor algorithm.

Problem 4.5.7 No hint available for this problem.

Problem 4.5.8 Apply de Boor's algorithm and use the differentiation formula.

4.6 Periodic Splines

Problem 4.6.1 Use Marsden's identity to obtain the relevant coefficients for $[0, 1]$ and apply the periodicity conditions.

Problem 4.6.2 Represent p on $[0, 1]$ as a linear combination of B-splines using Marsden's identity and determine α from the periodicity condition $c_{k+3} = c_k$.

5 Approximation

5.1 Schoenberg's Scheme

Problem 5.1.1 Show that $(Qf)' = \sum_k f'(x_k)\, b_{k,\xi}^{n-1}$ with $x_k \in [\xi_{k-1}^n, \xi_k^n]$.

Problem 5.1.2 Generalize the arguments for $\sqrt{x}$ using the ansatz $\xi_\ell = ((\ell-2)/(m-2))^\beta$ with an appropriate choice $\beta(\alpha)$.

Problem 5.1.3 Compare with the exact representation of x^2 via Marsden's identity.

Problem 5.1.4 Derive a lower bound for the relative error at the knot averages.

Problem 5.1.5 Write $f(x) = \sum_k f(x) b_{k,\xi}^n(x)$ and estimate $f(x) - f(\xi_k^n)$ using the uniform continuity of f on compact intervals.

5.2 Quasi-Interpolation

Problem 5.2.1 Use that $Qf = f$ for constant and linear functions.

Problem 5.2.2 Use symmetry, i.e., that $\alpha_0 = \gamma_2$, $\beta_0 = \beta_2$, $\gamma_0 = \alpha_2$, $\alpha_1 = \gamma_1$, to simplify the computations.

Problem 5.2.3 Test the identity $Q_k(\cdot - y)^2 = \psi_k(y)$ for $y = \eta_{k+\nu}$.

Problem 5.2.4 Use symmetry and make the ansatz $\gamma = \alpha$. Compare with Marsden's identity for $f(x) = 1$ and $f(x) = x^2$.

Problem 5.2.5 By symmetry, it is sufficient to construct Q_0. Test the identity $Q_0(\cdot - y)^3 = \psi_0(y)$ for $y = 0, h/2, \ldots, 2h$.

Problem 5.2.6 Testing the identity $Q_k(\cdot - y)^n = \psi_k(y)$ for $y = (k+\mu)h$ leads to an underdetermined linear system which is independent of h and k.

5.3 Accuracy of Quasi-Interpolation

Problem 5.3.1 Note that the B-spline coefficients do only change in the vicinity of newly added knots.

5.4 Stability

No problems for this section.

5.5 Interpolation

Problem 5.5.1 Choose $\xi_{j+2} = q^j$. To determine the matrix entries use the formulas for the first and last B-spline segment as well as the fact that the B-splines sum to one.

Problem 5.5.2 Use the formulas for the first and last B-spline segment and the fact that the B-splines sum to one. The not-a-knot condition yields two additional rows of the interpolation matrix which can be determined with the aid of the formulas for differentiating B-splines.

Problem 5.5.3 Note that for each interpolation point exactly n B-splines are relevant and that these B-splines vanish at all other points.

Problem 5.5.4 Use the ansatz $c_{k-2} = \gamma(\lambda^k + \lambda^{M-k})$ for the B-spline coefficients of the Lagrange spline.

5.6 Smoothing

Problem 5.6.1 Derive the orthogonality relation $\langle f'' - p'', b^1_{k,\xi} \rangle = 0$ which characterizes the orthogonal projection.

Problem 5.6.2 Consider the natural spline interpolants of a minimizing sequence f_ℓ.

Problem 5.6.3 Assume that $|f_k - p^\ell(x_k)| \geq \varepsilon$ for a sequence of smoothing splines with weights $w^\ell_k \to \infty$ and compare $E(p^\ell, \sigma)$ with the value for the interpolating natural spline.

6 Spline Curves

6.1 Control Polygon

Problem 6.1.1 Use multiple knots to model the corners in the letters β and γ.

Problem 6.1.2 No hint available for this problem.

6.2 Basic Properties

Problem 6.2.1 No hint available for this problem.

Problem 6.2.2 The Bernstein polynomials b_k^n correspond to the knot sequence $\tau_0 = 0, \ldots, 0 = \tau_n < \tau_{n+1} = 1, \ldots, 1 = \tau_{2n+1}$.

Problem 6.2.3 No hint available for this problem.

6.3 Refinement

Problem 6.3.1 Use the formulas of the B-splines on their first and last segment.

Problem 6.3.2 No hint available for this problem.

Problem 6.3.3 Use the estimate for the distance of a uniform spline curve from its control polygon in conjunction with the stability of the B-spline basis.

Problem 6.3.4 Compute the control points of the piecewise constant third derivative of the parametrization.

Problem 6.3.5 Insert successively the knots $h/2, 3h/2, 5h/2, \ldots$, and then simultaneously double their multiplicity.

Problem 6.3.6 No hint available for this problem.

Problem 6.3.7 Form a sufficiently large periodic extension of the control points.

6.4 Algorithms

Problem 6.4.1 Computing the point on the curve requires two, the tangent vector merely one knot insertion step.

Problem 6.4.2 No hint available for this problem.

Problem 6.4.3 Derive first a formula for $p'(\tau_k)$ and $p''(\tau_k)$ in terms of the control points c_{k-3}, c_{k-2}, and c_{k-1}, and then substitute the resulting expressions into the definition of curvature.

Problem 6.4.4 No hint available for this problem.

Problem 6.4.5 Determine first the equation of the hyperplane by solving an appropriate linear system.

Problem 6.4.6 No hint available for this problem.

6.5 Interpolation

Problem 6.5.1 Use the MATLAB function `ginput` for point specification as well as the AMB programs `spline_interpolation_cubic` for interpolating each coordinate separately with equally spaced abscissae and `spline_curve` for evaluation.

7 Multivariate Splines

7.1 Polynomials

Problem 7.1.1 Use the formulas for the derivatives at $x = (0, 0)$.

Problem 7.1.2 Interpolate successively with respect to the n coordinate directions, i.e., in the first step the values at $0 \times \{0, 1\}^{n-1}$ and $1 \times \{0, 1\}^{n-1}$ are interpolated based on the coordinate x_1.

7.2 Polynomial Approximation

Problem 7.2.1 Use that $\int (e - \varepsilon q)^2 < \int e^2$ for small ε if q has the same sign as the error e.

7.3 Splines

Problem 7.3.1 Use a subroutine which generates the values of the uniform univariate B-spline via the recurrence relation.

Problem 7.3.2 Consider the piecewise polynomial function restricted to lines parallel to $(1, -1)$. Use the minimal support property of B-splines to conclude that univariate splines with support on a single grid interval must vanish identically.

7.4 Algorithms

Problem 7.4.1 No hint available for this problem.

Problem 7.4.2 No hint available for this problem.

7.5 Approximation Methods

Problem 7.5.1 Process the array C successively in each of the coordinate directions.

Problem 7.5.2 Solve successively the univariate interpolation problems in each of the coordinate directions.

7.6 Hierarchical Bases

Problem 7.6.1 Subtract from the number of B-splines corresponding to a knot sequence in the tree Ξ the number of B-splines representable on finer grids.

Problem 7.6.2 Derive an explicit formula for the error of the linear interpolant.

8 Surfaces and Solids

8.1 Bézier Surfaces

Problem 8.1.1 By symmetry, the Bézier control points $c_{0,1}, c_{1,0}, c_{1,2}, c_{2,1}$ lie on the planes through the corresponding cube edges and the origin and can be determined using the endpoint interpolation property.

Problem 8.1.2 Construct a biquadratic patch containing three different line segments emerging from the origin.

Problem 8.1.3 Express the partial derivatives in terms of differences of the Bézier control points.

8.2 Spline Surfaces

Problem 8.2.1 The control points for approximating a circle form a square. Determine its diameter and rotate its position by the angles $\pi/2$, π and $3\pi/2$.

Problem 8.2.2 Parametrize the surface with bivariate splines of degree $(2, 1)$. To define the control points, move a gradually twisting line segment along a circle.

8.3 Subdivision Surfaces

Problem 8.3.1 Process rows and columns of the rectangular point array with the univariate scheme. Note that after each step the array boundaries have to be discarded.

Problem 8.3.2 For an irregular vertex p denote by e_ν the neighboring vertices on edges emerging from p and by v_ν the remaining vertices of the faces containing p. Set $E = \sum e_\nu, V = \sum v_\nu$ and determine the 3×3 matrix describing the modification of (p, E, V) by a subdivision step. The limit can now be determined by an eigenvalue analysis.

8.4 Blending

Problem 8.4.1 No hint available for this problem.

Problem 8.4.2 Store the patch values $p(s_k, t_\ell) \in \mathbb{R}^3$ as a list of arrays $p = \{p_1,\, p_2,\, p_3\}$.

8.5 Solids

Problem 8.5.1 Use the program `gausspar` to generate a grid of Gauß points and evaluate the parametrization with the program `spline_solid`.

9 Finite Elements

9.1 Ritz-Galerkin Approximation

Problem 9.1.1 Use the orthogonality of the basis functions and their derivatives:

$$\int_0^\pi \sin(jx)\sin(kx)\,dx = \int_0^\pi \cos(jx)\cos(kx)\,dx = \frac{\pi}{2}\,\delta_{j,k}$$

for $0 < j, k$.

Problem 9.1.2 In deriving the variational equations note that $\int_D \ldots = 2\pi \int_0^1 \ldots\, r\,dr$ for the unit disc D. Moreover, observe that the integrals $\int_0^1 \exp(r^2) B_k(r)\, r\,dr$ can be computed recursively.

Problem 9.1.3 By the mean value theorem, there exists a point x_k in every interval $D_k = kh + [0, h]$, such that $u'(x_k) - u_h'(x_k) = 0$. Use this fact to estimate $\int_{D_k} |u' - u_h'|^2$, noting that u_h' is constant on D_k with vanishing second order derivative.

9.2 Weighted B-Splines

Problem 9.2.1 Use the values of the scalar products of univariate B-splines and their derivatives. Moreover, note that $\sum_\ell g_{k,\ell} = 0$.

Problem 9.2.2 Use the MATLAB-commands `switch`, `strcat`, `fprintf`, `fopen`, and `fclose`.

9.3 Isogeometric Elements

Problem 9.3.1 Consider the images of the line segments $\{x\} \times [0, 1]$.

9.4 Implementation

Problem 9.4.1 The relevant bilinear B-splines are b_k, $k \in \{-1, 0\}^2$.

Problem 9.4.2 Determine the intersections of $\Gamma : 1 - 2xy = 0$ with the boundary of $(0, 1)^2$, and partition Ω accordingly. Map the tensor product Gauß formula to the resulting subdomains.

9.5 Applications

Problem 9.5.1 By the chain rule, $\partial_\nu r = x_\nu / r$.

Problem 9.5.2 The integrand is a constant which can be determined from the gradients of the hat-functions. Compute the gradients by considering appropriate directional derivatives.