ODE Text SAGE Examples

Example 1.1.10

```
A = matrix([[1,2,3],[4,5,6],[7,8,2]])
b = vector([-1,4,-7])
Aaug = A.augment(b); show(Aaug.rref())
```

$$\begin{pmatrix} 1 & 0 & 0 & \frac{139}{21} \\ 0 & 1 & 0 & -\frac{152}{21} \\ 0 & 0 & 1 & \frac{16}{7} \end{pmatrix}$$

Matrix algebra

```
A = matrix([[1,2,3],[4,-5,6]]); B = matrix([[-2,5,-8],[-7,9,3]]); C =
matrix([[-3,5],[2,1],[9,-7]])
a = vector([4,-7,9]); b = vector([2,-6,5])
```

```
AB = 2*A - 5*B; show(AB)
```

$$\begin{pmatrix} 12 & -21 & 46 \\ 43 & -55 & -3 \end{pmatrix}$$

```
ab = 4*a + 7*b; show(ab)
```

$$(30, -70, 71)$$

```
AC = A*C; show(AC)
```

$$\begin{pmatrix} 28 & -14 \\ 32 & -27 \end{pmatrix}$$

```
CA = C*A; show(CA)
```

$$\begin{pmatrix} 17 & -31 & 21 \\ 6 & -1 & 12 \\ -19 & 53 & -15 \end{pmatrix}$$

```
ACinv = AC.inverse
show(ACinv())
show(ACinv()*AC)
```

$$\begin{pmatrix} \frac{27}{308} & -\frac{1}{22} \\ \frac{8}{77} & -\frac{1}{11} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 1.4.5

```
A = matrix([[1,2,3],[-1,2,-3],[5,6,7]])
Ainv = A.inverse; show(Ainv())
```

$$\begin{pmatrix} -1 & -\frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{8} & -\frac{1}{8} \end{pmatrix}$$

Example 1.5.6

```
A = matrix([[3,4,7,-1],[2,6,8,-4],[-5,3,-2,-8],[7,-2,5,9]])
show(A.rref())
```

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 1.5.9

```
A = matrix([[3,4,-7,2],[2,6,9,-2],[-5,3,2,-13],[7,-2,5,16]])
b = vector([5,27,11,-1])
Aaug = A.augment(b); show(Aaug.rref())
```

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 1.7.9

```
A = matrix([[1,2,3,-3],[2,1,3,0],[1,-1,0,3],[-3,2,-1,-7]])
show(A.rref())
```

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 1.8.12

```
A = matrix([[1,2,3,4],[-1,2,1,0],[5,6,11,16],[2,4,6,8]])
show(A.rref())
```

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Compute the determinant

```
A = matrix([[-3,4,8,-2],[2,6,8,-4],[-5,-9,-2,-8],[7,-2,5,9]])
show(A)
show(A.determinant())
```

$$\begin{pmatrix} -3 & 4 & 8 & -2 \\ 2 & 6 & 8 & -4 \\ -5 & -9 & -2 & -8 \\ 7 & -2 & 5 & 9 \end{pmatrix}$$

$$-6624$$

Example 1.12.7

```
A = matrix([[1,2,3],[-1,2,-3],[5,6,7]])
A.eigenvectors_right()
```

```
[(-0.8576354035627644?, [(1, -0.1765540910982623?,
-0.5015090737887466?)], 1),
 (5.428817701781382? - 2.799969800175892?*I,
  [(1, -1.578389621117536? - 1.133436035853414?*I,
```

```
2.528532314672151? - 0.1776992428230211?*I)],
  1),
 (5.428817701781382? + 2.799969800175892?*I,
  [(1, -1.578389621117536? + 1.133436035853414?*I,
2.528532314672151? + 0.1776992428230211?*I)],
  1)]
```

Case Study 1.13.2

Eigenvalues and associated eigenvectors for **M**

```
M = matrix([[9/10,3/100,1/10],[2/100,85/100,2/10],[8/100,12/100,7/10]])
show(M.eigenvectors_right())
```

$$\left[\left(1, \left[\left(1,\ \frac{22}{21},\ \frac{24}{35}\right)\right], 1\right), \left(\frac{43}{50}, \left[\left(1,\ -\frac{6}{7},\ -\frac{1}{7}\right)\right], 1\right), \left(\frac{59}{100}, \left[(1,\ 3,\ -4)\right], 1\right)\right]$$

Rescaled eigenvector **v1**

```
show(vector([1,22/21,24/35])/(1+22/21+24/35))
```

$$\left(\frac{15}{41},\ \frac{110}{287},\ \frac{72}{287}\right)$$

Case Study 1.13.3

Eigenvalues and associated eigenvectors for **A**

```
A = matrix([[2/10,0,1/100],[8/10,5/10,0],[0,5/10,99/100]])
show(A.eigenvectors_right())
```

$$\left[\left(1, \left[\left(1,\ \frac{8}{5},\ 80\right)\right], 1\right), (0.2184101109882784?, [(1,\ -2.841011098827837?,\ 1.84$$

Rescaled eigenvector **v1**

```
show(vector([1,8/5,80])/(1+8/5+80))
```

$$\left(\frac{5}{413},\ \frac{8}{413},\ \frac{400}{413}\right)$$

Case Study 1.13.4

Eigenvalues and associated eigenvectors for **A**

```
A = matrix([[0,0,33/100],[18/100,0,0],[0,71/100,94/100]])
A.eigenvectors_right()
```

```
[(0.9835927397647997?, [(1, 0.1830025708028735?,
2.980584059893332?)], 1),
 (-0.02179636988239984? - 0.2059184804699735?*I,
  [(1, -0.09150128540143671? + 0.8644469584876894?*I,
-0.06604960570424193? - 0.6239953953635560?*I)],
  1),
 (-0.02179636988239984? + 0.2059184804699735?*I,
  [(1, -0.09150128540143671? - 0.8644469584876894?*I,
-0.06604960570424193? + 0.6239953953635560?*I)],
  1)]
```

Rescaled eigenvector **v1**

```
show(vector([1,0.1830025708028735,
2.980584059893332])/(1+0.1830025708028735+ 2.980584059893332))
```

$$\left(0.240177541311969,\, 0.0439531075092037,\, 0.715869351178827\right)$$

Solve **Pc** = **x0** by finding inverse of **P**

```
v1 = vector([0.240177541311969,0.0439531075092037,0.715869351178827])
v2 = vector([1,-0.09150128540143671-
0.8644469584876894*I,-0.06604960570424193+0.6239953953635560*I])
v3 =
vector([1,-0.09150128540143671+0.8644469584876894*I,-0.06604960570424193-
0.6239953953635560*I])
Ptr = matrix([v1,v2,v3]); P = Ptr.transpose()
Pinv = P.inverse; show(Pinv())
```

$$\left(\begin{array}{ll} 0.169504893926346 - 2.22044604925031 \times 10^{-16} i & 0. \\ 0.479644365668212 + 0.0464608524407570 i & -0.111231411900298 + 0.5 \\ 0.479644365668212 - 0.0464608524407570 i & -0.111231411900299 - 0.5 \end{array}\right.$$

Example 2.4.4

```
var("t")
a(t) = -exp(-3*t)*sin(2*t)
A(t) = a.integrate(t); show(A(t))
```

$$\frac{1}{13}\,(2\,\cos\left(2\,t\right) + 3\,\sin\left(2\,t\right))e^{\left(-3\,t\right)}$$

Example 2.5.4

```
var("t")
f(t) = t^2/(4+t); F(t) = f.integrate(t)
xp(t) = (4+t)*F(t); show(xp(t))
```

$$\frac{1}{2}\,\left(t^{2} - 8\,t + 32\,\log\left(t + 4\right)\right)\left(t + 4\right)$$

Example 2.5.11

```
var("t")
f(t) = t*sin(2*t)*exp(-5*t); F(t) = f.integrate(t)
xp(t) = exp(5*t)*F(t); show(xp(t))
```

$$-\frac{2}{841}\,\left(29\,t + 10\right)\cos\left(2\,t\right) - \frac{1}{841}\,\left(145\,t + 21\right)\sin\left(2\,t\right)$$

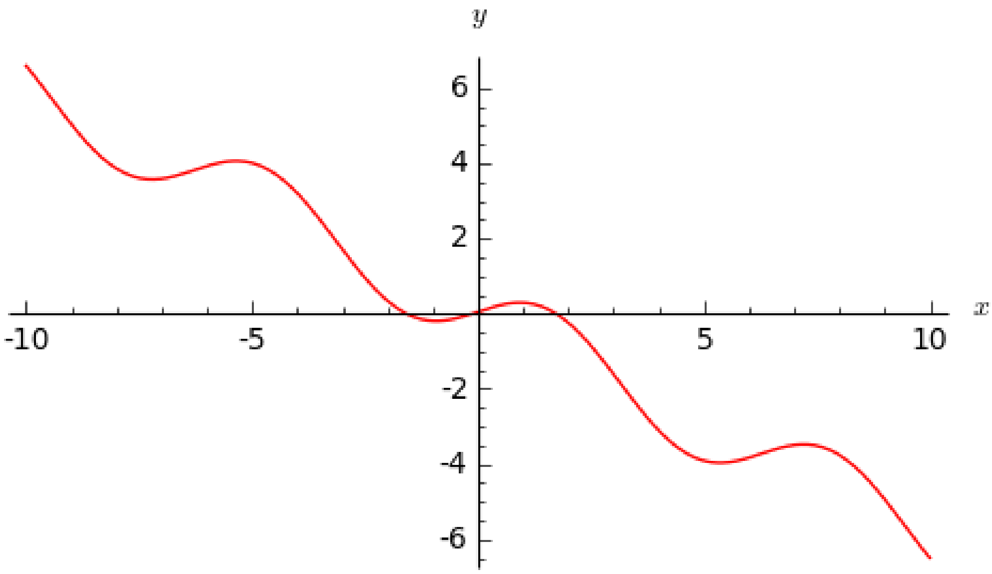
Case Study 2.6.1

```
var("a,c,t,w,V")
f(t) = exp(a*t)*(1-cos(w*t)); F(t) = f.integrate(t)
xp(t) = a*c*V*exp(-a*t)*F(t); show(factor(xp(t)))
```

$$-\frac{\left(a^{2}\cos\left(tw\right) + aw\sin\left(tw\right) - a^{2} - w^{2}\right)Vc}{a^{2} + w^{2}}$$

Exercise 2.6.9 - finding zeros of a function

```
f(x) = sin(x) - 0.6*x + 0.05
plotf = plot(f(x), (x,-10,10), color='red', thickness=1, axes_labels=
['$x$','$y$'])
show(plotf, figsize=[5,3])
```




```
find_root(f(x), -2, -1); find_root(f(x), -1, 0); f.find_root(1, 2, x)
```

```
-1.5832050218849731
-0.12582945316600264
1.7291476708245828
```

Case Study 3.7.1

```
var("a,c,V,w")
C = matrix([[-2*a,a,-w,0],[a,-2*a,0,-w],[w,0,-2*a,a],[0,w,a,-2*a]])
cf = vector([a*V*c,0,0,0])
cans = C\cf; cans
```

```
(-2/3*V*c + 2/3*V*c*w^2/((3*a + w^2/a)*a) - 2/3*(V*c*w + V*(3*a -
w^2/a)*c*w/(3*a + w^2/a))*((3*a - w^2/a)*w/((3*a + w^2/a)*a) +
w/a)/((3*a - w^2/a)^2/(3*a + w^2/a) - 12*a - 4*w^2/a), -1/3*V*c +
1/3*V*c*w^2/((3*a + w^2/a)*a) - 1/3*(V*c*w + V*(3*a -
w^2/a)*c*w/(3*a + w^2/a))*((3*a - w^2/a)*w/((3*a + w^2/a)*a) +
4*w/a)/((3*a - w^2/a)^2/(3*a + w^2/a) - 12*a - 4*w^2/a), -V*c*w/(3*a
+ w^2/a) + (V*c*w + V*(3*a - w^2/a)*c*w/(3*a + w^2/a))*(3*a -
w^2/a)/(((3*a - w^2/a)^2/(3*a + w^2/a) - 12*a - 4*w^2/a)*(3*a +
w^2/a)), 2*(V*c*w + V*(3*a - w^2/a)*c*w/(3*a + w^2/a))/((3*a -
w^2/a)^2/(3*a + w^2/a) - 12*a - 4*w^2/a))
```

Simplify each component of the solution vector

```
show(factor(cans[0]))
```

$$-\frac{2\,(3\,a^2+w^2)Va^2c}{(9\,a^2+w^2)(a^2+w^2)}$$

```
show(factor(cans[1]))
```

$$-\frac{\left(3\,a^2 - w^2\right)Va^2c}{(9\,a^2 + w^2)(a^2 + w^2)}$$

```
show(factor(cans[2]))
```

$$-\frac{\left(5\,a^2 + w^2\right)Vacw}{(9\,a^2 + w^2)(a^2 + w^2)}$$

```
show(factor(cans[3]))
```

$$-\frac{4\,Va^3cw}{(9\,a^2 + w^2)(a^2 + w^2)}$$

```
show(factor(C.determinant()))
```

$$\left(9\,a^2 + w^2\right)\left(a^2 + w^2\right)$$

Case Study 3.7.2

```
A = matrix([[-13/360,272/21875,7/200000],[1/90,-1/35,0],
[7/1800,0,-7/200000]])
A.eigenvectors_right()
```

```
[(-0.04468710144882567?, [(1, -0.6894599558852?, -0.0870930765?)],
1),
 (-0.01999979522243072?, [(1, 1.29626532758895?, -0.19478731665?)],
1),
 (-0.00003064301128330288?, [(1, 0.38930642212690?,
892.56345190559?)], 1)]
```

```
f = vector([49+3/10,0,0])
a0 = -A\f; show(a0)
```

$$\left(\frac{1848750}{1027}, \frac{2156875}{3081}, \frac{616250000}{3081}\right)$$

Case Study 3.7.3

```
A = matrix([[-1/2,1/80,73/80],[1/4,-961/80,0],[1/4,12,-73/80]])
A.eigenvectors_right()
```

```
[(0, [
  (1, 20/961, 38420/70153)
  ], 1),
 (-11.99123090808767?, [(1, 11.75414545343074?,
```

```
-12.75414545343074?)], 1),
 (-1.433769091912338?, [(1, 0.02363232434704146?,
-1.023632324347042?)], 1)]
```

```
show(vector([1,20/961,38420/70153])/(1+20/961+38420/70153))
```

$$\left(\frac{70153}{110033}, \frac{1460}{110033}, \frac{38420}{110033}\right)$$

```
A = matrix([[-1,1/80,1/16],[1/2,-961/80,0],[1/2,12,-1/16]])
A.eigenvectors_right()
```

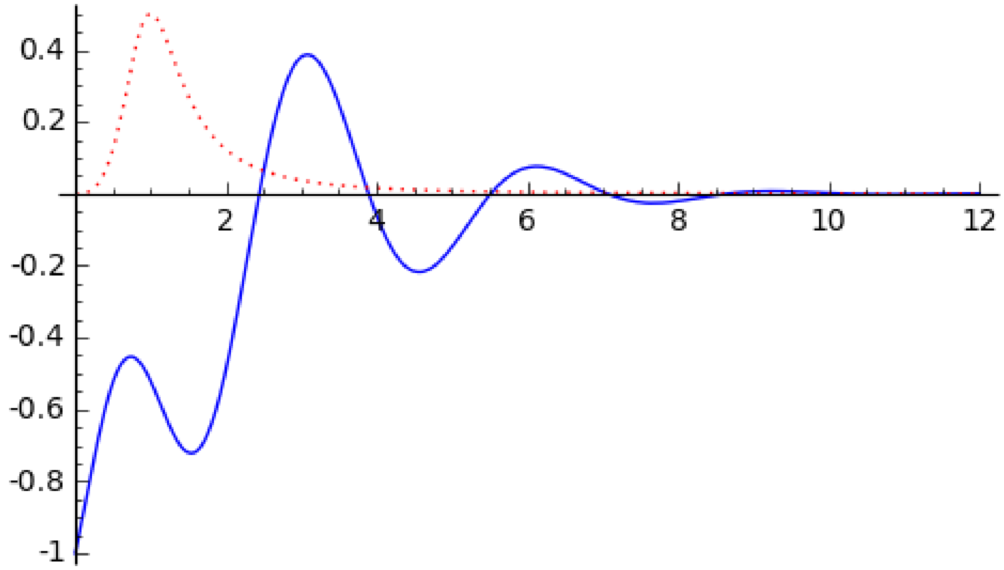
```
[(0, [
  (1, 40/961, 15368/961)
  ], 1),
 (-12.010216418744899?, [(1, 218.9543283748980?,
-219.9543283748980?)], 1),
 (-1.064783581255101?, [(1, 0.04567162510200666?,
-1.045671625102007?)], 1)]
```

```
show(vector([1,40/961,15368/961])/(1+40/9615368/961))
```

$$\left(\frac{961}{16369}, \frac{40}{16369}, \frac{15368}{16369}\right)$$

Plot multiple functions on one graph

```
f(x) = (x^2*cos(2*x)-1)*exp(-x)
g(x) = x^3/(1+x^6)
plotf = plot(f(x), (x,0,12), color='blue', linestyle='-', thickness=1)
plotg = plot(g(x), (x,0,12), color='red', linestyle=':', thickness=1)
plotall = plotf + plotg
show(plotall, figsize=[5,3])
```



Example 4.1.3 - roots of a polynomial will be found by finding the eigenvalues of the companion matrix for the ODE

```
A = matrix([[0,1,0,0],[0,0,1,0],[0,0,0,1],[-9,-6,3,0]])
show(A)
A.eigenvalues()
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -6 & 3 & 0 \end{pmatrix}$$

```
[-1.919039750549684?, -1.132826740049513?, 1.525933245299598? -
1.345911313085324?*I, 1.525933245299598? + 1.345911313085324?*I]
```

Example 5.2.10

```
var("s,t"); assume(s>0)
f1 = exp(-s*t)*t; f2 = exp(-s*t)*(t^2-2)
F1 = integral(f1,t,0,2); F2 = integral(f2,t,2,infinity); F = F1+F2
show(factor(F))
```

$$\frac{\left(se^{(2\,s)} + 3\,s + 2\right)e^{(-2\,s)}}{s^3}$$

Example 5.2.13

```
F = (s^4+6*s^3-15*s^2-10*s+50)/(s^3*(s^2-2*s+10))
```

```
show(F.partial_fraction())
```

$$\frac{3\,s + 2}{s^{2} - 2\,s + 10} - \frac{2}{s} + \frac{5}{s^{3}}$$