

Contents

Preface	xiii
I Coming to Grips with Sage	1
1 First Steps	3
1.1 The Sage Program	3
1.1.1 A Tool for Mathematics	3
1.2 Sage as a Calculator	7
1.2.1 First Computations	7
1.2.2 Elementary Functions and Usual Constants	10
1.2.3 Online Help and Automatic Completion	11
1.2.4 Python Variables	12
1.2.5 Symbolic Variables	13
1.2.6 First Graphics	15
2 Analysis and Algebra	17
2.1 Symbolic Expressions and Simplification	17
2.1.1 Symbolic Expressions	17
2.1.2 Transforming Expressions	18
2.1.3 Usual Mathematical Functions	20
2.1.4 Assumptions	21
2.1.5 Some Pitfalls	22
2.2 Equations	23
2.2.1 Explicit Solving	23
2.2.2 Equations with No Explicit Solution	26
2.3 Analysis	27
2.3.1 Sums	27
2.3.2 Limits	28
2.3.3 Sequences	28
2.3.4 Power Series Expansions	30
2.3.5 Series	31
2.3.6 Derivatives	32
2.3.7 Partial Derivatives	33
2.3.8 Integrals	33
2.4 Basic Linear Algebra	35
2.4.1 Solving Linear Systems	35

2.4.2	Vector Computations	35
2.4.3	Matrix Computations	36
2.4.4	Reduction of a Square Matrix	37
3	Programming and Data Structures	39
3.1	Syntax	39
3.1.1	General Syntax	39
3.1.2	Function Calls	41
3.1.3	More about Variables	41
3.2	Algorithmics	42
3.2.1	Loops	42
3.2.2	Conditionals	49
3.2.3	Procedures and Functions	50
3.2.4	Example: Fast Exponentiation	53
3.2.5	Input and Output	56
3.3	Lists and Other Data Structures	57
3.3.1	List Creation and Access	57
3.3.2	Global List Operations	59
3.3.3	Main Methods on Lists	63
3.3.4	Examples of List Manipulations	65
3.3.5	Character Strings	66
3.3.6	Shared or Duplicated Data Structures	67
3.3.7	Mutable and Immutable Data Structures	68
3.3.8	Finite Sets	69
3.3.9	Dictionaries	70
4	Graphics	73
4.1	2D Graphics	73
4.1.1	Graphical Representation of a Function	73
4.1.2	Parametric Curve	76
4.1.3	Curve in Polar Coordinates	76
4.1.4	Curve Defined by an Implicit Equation	77
4.1.5	Data Plot	77
4.1.6	Displaying Solutions of Differential Equations	80
4.1.7	Evolute of a Curve	86
4.2	3D Graphics	88
5	Computational Domains	93
5.1	Sage Is Object Oriented	93
5.1.1	Objects, Classes, and Methods	93
5.1.2	Objects and Polymorphism	95
5.1.3	Introspection	96
5.2	Elements, Parents, Categories	97
5.2.1	Elements and Parents	97
5.2.2	Constructions	98
5.2.3	Further Reading: Categories	99

5.3	Domains with a Normal Form	99
5.3.1	Elementary Domains	101
5.3.2	Compound Domains	105
5.4	Expressions vs Computational Domains	107
5.4.1	Symbolic Expressions as a Computational Domain	107
5.4.2	Examples: Polynomials and Normal Forms	107
5.4.3	Example: Polynomial Factorization	108
5.4.4	Synthesis	110
II Algebra and Symbolic Computation		111
6	Finite Fields and Number Theory	113
6.1	Finite Fields and Rings	113
6.1.1	The Ring of Integers Modulo n	113
6.1.2	Finite Fields	115
6.1.3	Rational Reconstruction	116
6.1.4	The Chinese Remainder Theorem	117
6.2	Primality	118
6.3	Factorization and Discrete Logarithms	121
6.4	Applications	122
6.4.1	The Constant δ	122
6.4.2	Computation of a Multiple Integral	123
7	Polynomials	125
7.1	Polynomial Rings	126
7.1.1	Introduction	126
7.1.2	Building Polynomial Rings	126
7.1.3	Polynomials	128
7.2	Euclidean Arithmetic	132
7.2.1	Divisibility	132
7.2.2	Ideals and Quotients	134
7.3	Factorization and Roots	135
7.3.1	Factorization	135
7.3.2	Root Finding	137
7.3.3	Resultant	138
7.3.4	Galois Group	140
7.4	Rational Functions	140
7.4.1	Construction and Basic Properties	140
7.4.2	Partial Fraction Decomposition	141
7.4.3	Rational Reconstruction	142
7.5	Formal Power Series	145
7.5.1	Operations on Truncated Power Series	146
7.5.2	Solutions of an Equation: Series Expansions	147
7.5.3	Lazy Power Series	148
7.6	Computer Representation of Polynomials	149

8	Linear Algebra	153
8.1	Elementary Constructs and Manipulations	153
8.1.1	Spaces of Vectors and Matrices	153
8.1.2	Vector and Matrix Construction	155
8.1.3	Basic Manipulations and Arithmetic on Matrices	156
8.1.4	Basic Operations on Matrices	158
8.2	Matrix Computations	158
8.2.1	Gaussian Elimination, Echelon Form	159
8.2.2	Linear System Solving, Image and Nullspace Basis	166
8.2.3	Eigenvalues, Jordan Form, and Similarity Transformation	167
9	Polynomial Systems	177
9.1	Polynomials in Several Variables	177
9.1.1	The Rings $A[x_1, \dots, x_n]$	177
9.1.2	Polynomials	179
9.1.3	Basic Operations	180
9.1.4	Arithmetic	181
9.2	Polynomial Systems and Ideals	182
9.2.1	A First Example	182
9.2.2	What Does Solving Mean?	185
9.2.3	Ideals and Systems	185
9.2.4	Elimination	190
9.2.5	Zero-Dimensional Systems	196
9.3	Gröbner Bases	200
9.3.1	Monomial Orders	201
9.3.2	Division by a Family of Polynomials	202
9.3.3	Gröbner Bases	203
9.3.4	Gröbner Basis Properties	206
9.3.5	Computations	209
10	Differential Equations and Recurrences	213
10.1	Differential Equations	213
10.1.1	Introduction	213
10.1.2	First-Order Ordinary Differential Equations	214
10.1.3	Second-Order Equations	221
10.1.4	The Laplace Transform	223
10.1.5	Systems of Linear Differential Equations	224
10.2	Recurrence Relations	226
10.2.1	Recurrences $u_{n+1} = f(u_n)$	226
10.2.2	Linear Recurrences with Rational Coefficients	229
10.2.3	Non-Homogeneous Linear Recurrence Relations	229

III Numerical Computation	231
11 Floating-Point Numbers	233
11.1 Introduction	233
11.1.1 Definition	233
11.1.2 Properties and Examples	234
11.1.3 Standardization	234
11.2 The Floating-Point Numbers	235
11.2.1 Which Kind of Floating-Point Numbers to Choose?	236
11.3 Properties of Floating-Point Numbers	237
11.3.1 These Sets Are Full of Gaps	237
11.3.2 Rounding	238
11.3.3 Some Properties	238
11.3.4 Complex Floating-Point Numbers	243
11.3.5 Methods	244
11.4 Interval and Ball Arithmetic	244
11.4.1 Implementation in Sage	245
11.4.2 Computing with Real Intervals and Real Balls	248
11.4.3 Some Examples of Applications	249
11.4.4 Complex Intervals and Complex Balls	251
11.4.5 Usage and Limitations	252
11.4.6 Interval Arithmetic Is Used by Sage	252
11.5 Conclusion	252
12 Non-Linear Equations	255
12.1 Algebraic Equations	255
12.1.1 The Method <code>Polynomial.roots()</code>	255
12.1.2 Representation of Numbers	256
12.1.3 The Fundamental Theorem of Algebra	257
12.1.4 Distribution of the Roots	257
12.1.5 Solvability in Radicals	258
12.1.6 The Method <code>Expression.roots()</code>	260
12.2 Numerical Solution	261
12.2.1 Location of Solutions of Algebraic Equations	262
12.2.2 Iterative Approximation Methods	263
13 Numerical Linear Algebra	277
13.1 Inexact Computations	277
13.1.1 Matrix Norms and Condition Number	278
13.2 Dense Matrices	281
13.2.1 Solving Linear Systems	281
13.2.2 Direct Resolution	281
13.2.3 The <i>LU</i> Decomposition	282
13.2.4 The Cholesky Decomposition	283
13.2.5 The <i>QR</i> Decomposition	284
13.2.6 Singular Value Decomposition	284
13.2.7 Application to Least Squares	285

13.2.8	Eigenvalues, Eigenvectors	288
13.2.9	Polynomial Curve Fitting: The Devil Is Back	293
13.2.10	Implementation and Efficiency	296
13.3	Sparse Matrices	297
13.3.1	Where Do Sparse Systems Come From?	297
13.3.2	Sparse Matrices in Sage	298
13.3.3	Solving Linear Systems	298
13.3.4	Eigenvalues, Eigenvectors	300
13.3.5	More Thoughts on Solving Large Non-Linear Systems	301
14	Numerical Integration	303
14.1	Numerical Integration	303
14.1.1	Available Integration Functions	309
14.1.2	Multiple Integrals	315
14.2	Solving Differential Equations	316
14.2.1	An Example	317
14.2.2	Available Functions	319
IV	Combinatorics	323
15	Enumeration and Combinatorics	325
15.1	Initial Examples	326
15.1.1	Poker and Probability	326
15.1.2	Enumeration of Trees Using Generating Functions	328
15.2	Common Enumerated Sets	334
15.2.1	First Example: Subsets of a Set	334
15.2.2	Integer Partitions	336
15.2.3	Some Other Finite Enumerated Sets	338
15.2.4	Set Comprehension and Iterators	341
15.3	Constructions	347
15.4	Generic Algorithms	349
15.4.1	Lexicographic Generation of Lists of Integers	349
15.4.2	Integer Points in Polytopes	351
15.4.3	Species, Decomposable Combinatorial Classes	351
15.4.4	Objects up to Isomorphism	353
16	Graph Theory	361
16.1	Constructing Graphs	361
16.1.1	Starting from Scratch	361
16.1.2	Available Constructors	363
16.1.3	Disjoint Unions	366
16.1.4	Graph Visualization	367
16.2	Methods of the <code>Graph</code> Class	370
16.2.1	Modification of Graph Structure	370
16.2.2	Operators	371
16.2.3	Graph Traversal and Distances	372

16.2.4	Flows, Connectivity, Matching	373
16.2.5	NP-Complete Problems	374
16.2.6	Recognition and Testing of Properties	375
16.3	Graphs in Action	377
16.3.1	Greedy Vertex Coloring of a Graph	377
16.3.2	Generating Graphs under Constraints	379
16.3.3	Find a Large Independent Set	380
16.3.4	Find an Induced Subgraph in a Random Graph	381
16.4	Some Problems Solved Using Graphs	383
16.4.1	A Quiz from the French Journal “Le Monde 2”	383
16.4.2	Task Assignment	384
16.4.3	Plan a Tournament	385
17	Linear Programming	387
17.1	Definition	387
17.2	Integer Programming	388
17.3	In Practice	388
17.3.1	The <code>MixedIntegerLinearProgram</code> Class	388
17.3.2	Variables	389
17.3.3	Infeasible or Unbounded Problems	390
17.4	First Applications in Combinatorics	391
17.4.1	Knapsack	391
17.4.2	Matching	392
17.4.3	Flow	393
17.5	Generating Constraints and Application	395
	Appendix: Answers to Exercises	401
1	First Steps	401
2	Analysis and Algebra	401
4	Graphics	409
5	Computational Domains	413
6	Finite Fields and Number Theory	414
7	Polynomials	419
8	Linear Algebra	423
9	Polynomial Systems	425
10	Differential Equations and Recurrences	428
11	Floating-Point Numbers	429
12	Non-Linear Equations	433
13	Numerical Linear Algebra	435
14	Numerical Integration	436
15	Enumeration and Combinatorics	438
16	Graph Theory	444
17	Linear Programming	445
	Bibliography	447
	Index	451