

# Preface

Our times can be characterized by, among many other attributes, the seemingly increasing speed of everything. Within science, it has led to the publication explosion, which reflects the fact that, using the words of Cornelius Lanczos in his essay *The Inspired Guess in The History of Physics*, published in 1964 [119], “science is big business.” Under the present circumstances it therefore has become increasingly difficult to stop and invest time and energy into collaboration across disciplines.

The authors of this text work in different fields, namely, in mathematical modeling and PDE analysis and in computational mathematics with emphasis on matrix computations. Given the overspecialization of mathematics (and science in general), the probability that they meet at a conference or workshop and would bounce ideas between themselves is generally very low. Fortunately for them, this did happen a couple of years ago, and the present text is one of the consequences. They found working on it useful and enjoyable, with the goal of finding a common language and developing a mutual understanding for each other’s work. In this way they have reacted to the feeling which they had during their first professional discussion and which is so clearly expressed in the beautiful, poetic words of the *one true geometrician in the empire* in *The Wisdom of the Sands* by Antoine de Saint-Exupéry [173, Chapter 115]:

*I would not bid you pore upon a heap of stones, and turn them over and over, in the vain hope of learning from them the secret of meditation. For on the level of the stones there is no question of meditation; for that, the temple must have come into being. But, once it is built, a new emotion sways my heart, and when I go away, I ponder on the relations between the stones. ...*

*I must begin by feeling love; and I must first observe a wholeness. After that I may proceed to study the components and their groupings. But I shall not trouble to investigate these raw materials unless they are dominated by something on which my heart is set. Thus I began by observing the triangle as a whole; then I sought to learn in it the functions of its component lines. ...*

*So, to begin with, I practise contemplation. After that, if I am able, I analyse and explain. ...*

*Little matter the actual things that are linked together; it is the links that I must begin by apprehending and interpreting.*

The authors present the result of their effort to others in the hope that at least some part of it could be useful in strengthening common understanding among communities dealing with mathematics of problems that can be encountered on the way from the formulation of the mathematical model and its analysis towards numerical computation of the discretized systems of algebraic equations.

The case study which is used throughout the text is built around the unifying principle of minimizing the energy of a given infinite-dimensional system as well as the associated

discretized finite-dimensional system. That naturally links a class of linear elliptic partial differential boundary value problems described later with the well-known conjugate gradient method, which can be derived using an appropriate Hilbert space setting and then discretized in order to obtain its algebraic formulation. Moreover, the conjugate gradient method can be thought of as a model reduction procedure producing a sequence of discrete models of increasing size which match moments of the original infinite-dimensional model. A more detailed description with a chapter-by-chapter overview of this little book will be presented in the introduction with the help of some basic notation given there.

Our approach can perhaps be characterized by the following quotes. Two of them are taken, as the quote above, from the work of Cornelius Lanczos, who personified in a brilliant way the unity of analytic and computational views (and, in addition, an outstanding unity of knowledge of mathematics and physics as well as an extraordinary depth of philosophical thought). The first one points out the need for mutual understanding, and it is taken from the preface of the monograph *Linear Differential Operators*, published in 1961 [120]:

*To get an explicit solution of a given boundary value problem is in this age of large electronic computers no longer a basic question. The problem can be coded for the machine and the numerical answer obtained. But of what value is the numerical answer if the scientist does not understand the peculiar analytical properties and idiosyncrasies of the given operator?*

The second quote resonates with the authors' personal feeling that it is good to stop for a while and contemplate in order to understand what has been achieved or, perhaps, to give a second thought to some commonly accepted and widely communicated views. It is taken from the essay [119] of Lanczos mentioned above:

*Once the great mathematician Gauss was engaged in a particularly important investigation, but seemed to make little headway. His colleagues inquired when the publication was to appear. Gauss gave them an apparently paradoxical and yet perfectly correct answer: "I have all the results but I don't know yet how I am going to get them."*

Finally, we have found very relevant to point out the following quote from the seminal paper of John von Neumann and Herman H. Goldstine [191] that is often considered as the beginning of the field of numerical analysis:

*When a problem in pure or in applied mathematics is "solved" by numerical computation, errors, that is, deviations of the numerical "solution" obtained from the true, rigorous one, are unavoidable. Such a "solution" is therefore meaningless, unless there is an estimate of the total error in the above sense.*

*Such estimates have to be obtained by a combination of several different methods, because the errors that are involved are aggregates of several different kinds of contributory, primary errors. These primary errors are so different from each other in their origin and character, that the methods by which they have to be estimated must differ widely from each other. A discussion of the subject may, therefore, advantageously begin with an analysis of the main kinds of primary errors, or rather of the sources from which they spring.*

*This analysis of the sources of errors should be objective and strict inasmuch as completeness is concerned, ....*

The investigation of *all sources of errors* should be indeed an inherent part of any numerical computation. We illustrate this point within the last chapters of the text.

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The text is as self-contained as possible, but it is inevitably restrictive. We would be grateful for any feedback concerning its contents and presentation as well as for possible comments which can arise from using it as teaching material.

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