

Chapter 1

Dimensional Analysis and Scaling

1.1 Mathematical models

A mathematical model describes the behavior of a real-life system in terms of mathematical equations. These equations represent the relations between the relevant properties of the system under consideration. In these models we meet with *variables* and *parameters*. In variables, we discern between *dependent* and *independent*. For example, in mechanical systems one usually is interested in the positions of the different parts as functions of time, so in these systems the positions act as the dependent variables and time as the independent variable. Parameters are properties like masses, prescribed temperatures, currents, voltages, and friction coefficients. Parameters that can be influenced by the observer are referred to as *adjustable*. The other parameters act as constants in the model. For example, in atmospheric models used in weather forecasting one is interested in properties like temperature and humidity (the dependent variables) as functions of position and time (the independent variables). Important parameters are then the gravity field and the rotational speed of the earth, and these clearly belong to the class of *nonadjustable* parameters. The *solution* of a mathematical model is known if we can determine the relations between dependent and independent variables. Since the solution depends on the values of the adjustable parameters, mathematical models are a powerful tool with which to determine which values of the adjustable parameters yield specific required behavior.

1.2 Dimensions

If the variables or parameters in a model correspond to physical properties, they have physical dimensions. The *fundamental dimensions* used in this book are given in the following table:

dimension	symbol	MKS-unit
length	L	m (meter)
mass	M	kg (kilogram)
time	T	s (second)
temperature	Θ	$^{\circ}C$ (degree Celsius)
current	I	A (Ampere)

The dimension of any physical quantity can be expressed in terms of the fundamental dimensions. For most quantities this is clear from the definition. For example,

quantity	dimension
area	L^2
volume	L^3
velocity	L/T
acceleration	L/T^2
mass density	M/L^3
mechanical energy	ML^2/T^2
pressure	$M/(LT^2)$

In other cases the dimensionality of a quantity is deduced from the rule that all terms in a particular equation must have the same dimensionality. This rule is a consequence of the condition that the form of any equation in a mathematical model may not depend on the units used. For example, the dimension of force directly follows from the second law of Newton, which states that for a single mass, the mass times the acceleration equals the total force exerted on the mass. In standard notation, $F = ma$. So, the dimensionality of a force F , denoted as $[F]$, equals the dimensionality $[ma]$ of the product of mass m and acceleration a . Since $[ma] = [m][a]$, we conclude that $[F] = ML/T^2$. In this way we arrive at, e.g.,

quantity	dimension
force	ML/T^2
mechanical energy	ML^2/T^2
pressure	$M/(LT^2)$

For coefficients, the dimensionality may vary with specific choices made by the modeler. For example, if a frictional force is introduced with strength proportional to the velocity of the object, the constant of proportionality will have the dimension of the quotient of force and velocity. However, if the friction is assumed to be proportional to the velocity squared, the proportionality constant will have the dimension of force divided by velocity squared. See also Example 1.3c. For a *dimensionless quantity*, say, q , we have $[q] = 1$. Examples are angles and universal constants like π and e . In dimensional analysis, to be treated in §1.3, dimensionless quantities play a central role. There it is shown how they can be constructed. The existence of these so-called dimensionless numbers allows us to draw important conclusions about the system without solving the governing mathematical model.

Example 1.2a. Driven, damped, harmonic oscillator.

Consider a bead-spring system in one dimension under the influence of friction and a driving force. The position of the bead with mass m is denoted by its displacement u measured with respect to its equilibrium position. See Fig. 1.1.

We are interested in u as a function of time t . So, u is the dependent and t the independent variable of the system. As for the notation, we shall use the convention $\dot{u} \equiv du/dt$, $\ddot{u} \equiv d^2u/dt^2$, etc. The second law of Newton states that the *inertia force*, i.e., mass m times acceleration \ddot{u} , equals the sum of the forces exerted on the bead. These forces are the driving force F_d , which is taken harmonic with angular frequency ω and amplitude F_0 ;

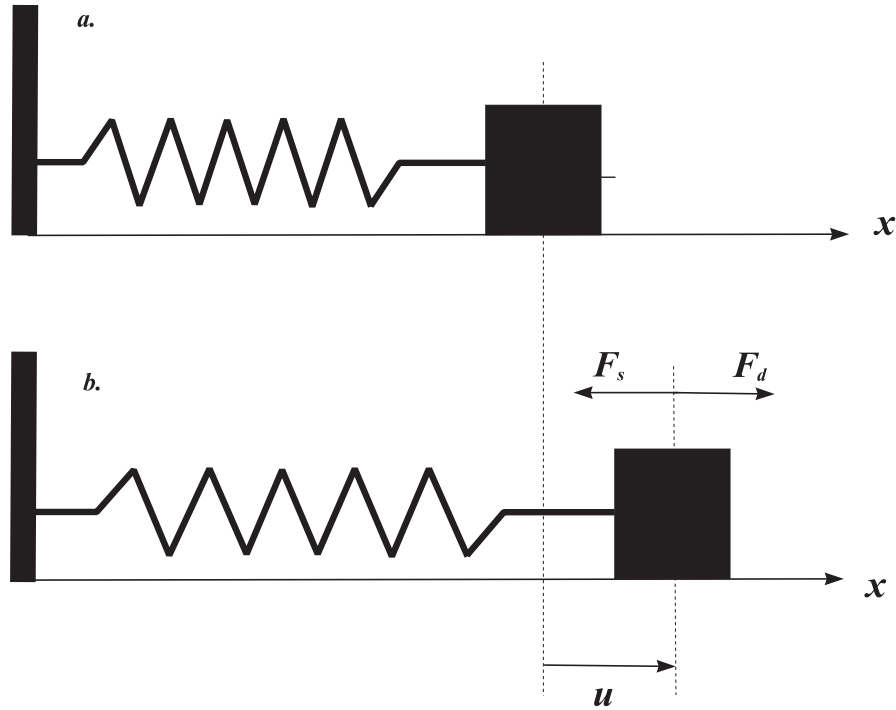


Figure 1.1. A vibrating mass attached to a spring is the prototype of harmonic motion if the spring response is linear, i.e., if the spring force F_s is linearly proportional to the deviation u (b) measured with respect to the equilibrium position (a). Apart from F_s , often an external driving force F_d and a friction force F_f apply. The latter usually is taken linearly proportional to the velocity of the mass but in the reverse direction.

the spring force F_s , which is linearly proportional to the displacement and reversely directed to it; and the frictional force F_f , which is assumed to be linearly proportional to the velocity and reversely directed to it. This leads to the balance of forces

$$m\ddot{u} = F_f + F_s + F_d = -c\dot{u} - ku + F_0 \sin \omega t.$$

The conventional form in which to write this *equation of motion* is

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \omega t. \tag{1.1}$$

Since m , c , k , and F_0 can all be influenced, they are adjustable parameters. Every term in this equation has the dimension of force, so ML/T^2 . From this it follows that

$$[c] = \frac{ML/T^2}{L/T} = \frac{M}{T}, \quad [k] = \frac{ML/T^2}{L} = \frac{M}{T^2}, \quad [F_0] = \frac{ML}{T^2}.$$

The argument ωt of the sine function must be dimensionless, so $[\omega t] = 1$. We thus have

$$[\omega] = \frac{1}{T}.$$

A mathematical model in terms of an ordinary differential equation (ODE) is not yet complete if the initial values are left unspecified. This introduces two extra parameters into the system: initial position $u_0 \equiv u(t_0)$ and initial velocity $v_0 \equiv \dot{u}(t_0)$. The solution $u(t)$ thus depends on seven parameters, and we could write it as

$$u = u(t; m, c, k, F_0, \omega, u_0, v_0).$$

For such a simple system this is a huge number to handle, since in an experiment all these parameters could in principle be varied. In the following we show that such a system can essentially be described with fewer parameters, since it does not make sense to vary them all independently. \square

The fact that the variables and parameters have physical dimensions can be fruitfully exploited. The techniques of *nondimensionalizing* and *scaling* are extremely powerful tools in analyzing the models. Their importance is fully appreciated only through examples, which account for the largest part of this chapter. The basic idea is to apply a transformation to the variables and parameters such that simplified equations result. It is often amazing how much structure is revealed simply by nondimensionalizing, without solving the model explicitly. Thanks to these techniques it is often known beforehand that the system depends not on all parameters separately but only on certain combinations. In an experimental situation it is of great importance to know how the system depends on the parameters, so this insight may save much time, cost, and energy.

In practice two methods are applied, *dimensional analysis* and *scaling*, each having its own merits. They are dealt with in the subsections below, respectively. Dimensional analysis fully exploits the information contained in the physical dimensions of the variables and parameters. Scaling has a more restricted scope and aims at a reduction of the number of parameters.

1.3 Dimensional analysis

Nondimensionalizing a mathematical model is a constructive way to formulate the model in terms of dimensionless quantities only. A big achievement is that dimensional analysis yields insight in the scaling relations of the system without using knowledge of any governing equation. An advantageous corollary is that the total number of variables and/or parameters is minimal. Reduction of the number of parameters is also the purpose of *scaling*, a technique to be dealt with in the next section. However, dimensional analysis is more general than scaling in that it is based on a transformation of both variables and parameters on the same footing, whereas in scaling only the variables are transformed. Another difference is that scaling starts from the governing equations, whereas dimensional analysis starts much more basically, namely, from the dimensions involved in the system, and it may even predict from them some quantitative features of the model without knowledge of the model equations. The basic idea of dimensional analysis is easily explained. Consider a system with scalar variables x_1, \dots, x_k and scalar parameters p_1, \dots, p_ℓ . So, the total number of quantities involved is $N = k + \ell$. Note that in the model, vectors, matrices, etc., may figure, but for this analysis all their components have to be treated separately. We now form the products

$$x_1^{r_1} \dots x_k^{r_k}, p_1^{r_{k+1}} \dots p_\ell^{r_N}$$

and ask for which choices of the r_i these products are dimensionless. The answer follows from replacing each x_i and p_i with its fundamental dimensions. If, say, m dimensions d_1, \dots, d_m are involved, the replacement gives rise to another type of product,

$$d_1^{s_1} \dots d_m^{s_m},$$

with the numbers $s_i, i = 1, \dots, m$, being linear functions of the $r_j, j = 1, \dots, N$. The procedure is illustrated several times in the examples below. By requiring

$$s_i = 0, \quad i = 1, \dots, m,$$

we obtain a set of m linear equations for the N unknowns r_1, \dots, r_N . Note that the numbers $r_j, j = 1, \dots, N$, are rational, since they are solutions of linear equations with rational coefficients. The rationality of these coefficients stems from the fact that in nature all measurable quantities turn out to have dimensions that are products of integer powers of the fundamental dimensions, as shown in the tables in §1.2. From linear algebra it follows that there are (at most) $N - m$ linearly independent solutions, corresponding to $N - m$ dimensionless quantities $q_i, i = 1, \dots, (N - m)$. Buckingham formalized this in the following theorem.

Theorem (Buckingham).

Consider a system with variables x_1, \dots, x_k and parameters p_1, \dots, p_ℓ , in which m fundamental dimensions are involved. Then, $k + \ell - m$ dimensionless quantities q_i can be defined, which are products and quotients of the original variables and parameters. Each (scalar) model equation

$$f(x_1, \dots, x_k, p_1, \dots, p_\ell) = 0$$

between the x_i and p_i of a mathematical model can be replaced with a corresponding relation between the q_i :

$$f^*(q_1, \dots, q_{k+\ell-m}) = 0.$$

Since Buckingham [6] denoted the dimensionless quantities by π_i , this theorem is often referred to as the π -theorem of Buckingham. We shall not follow his notation since it is no longer common in the literature. As follows from the construction of the q_i as solutions of an underdetermined set of linear equations, they are not uniquely defined by the procedure. If the procedure yields a set of q_i , we can apply a transformation, e.g., by taking algebraic or even functional combinations of them, obtaining another set of dimensionless quantities of the system. It is a matter of expertise, and partly of taste, to determine a convenient set of q_i for the system under consideration. If the number of variables and parameters is not small, the freedom of choice must be especially exploited with care.

We shall work out the nondimensionalizing procedure for a considerable number of examples, pointing out both the practical aspects of the technique and the insight it may yield about the behavior of the system without solving the equations explicitly.

Example 1.3a. Catapulting.

Let us start with an example in which the mathematics is very basic but the ideas behind dimensional analysis are clearly illustrated. A projectile with mass m is launched vertically. See Fig. 1.2. At launching it has velocity v_0 . Its trajectory, i.e., its vertical position z as a function of time t is assumed to be completely determined by the influence of gravity. The effect of friction due to the air is ignored here (but dealt with in Example 1.3e). The projectile will decelerate because of gravity until it reaches its highest position z_{\max} at time t_{\max} . After that it falls back with increasing velocity and arrives on the earth at time t_{final} . Since we take v_0 such that z_{\max} remains small compared to the Earth's radius, we may take the gravity field uniform with gravity constant g .

In this system the variables are z and t and the parameters are m , v_0 , and g . The relevant physical dimensions are M , L , and T . So, $k = 2$, $\ell = 3$, and $m = 3$, and the theorem of Buckingham states that the system has two dimensionless quantities. All properties of the system can be expressed in only these two quantities. In this simple case the dimensionless quantities can be easily found from inspection of the dimensions: $[z] = L$, $[t] = T$, $[m] = M$, $[v_0] = L/T$, and $[g] = L/T^2$. An evident choice is

$$t^* = \frac{gt}{v_0}, \quad z^* = \frac{gz}{v_0^2}.$$

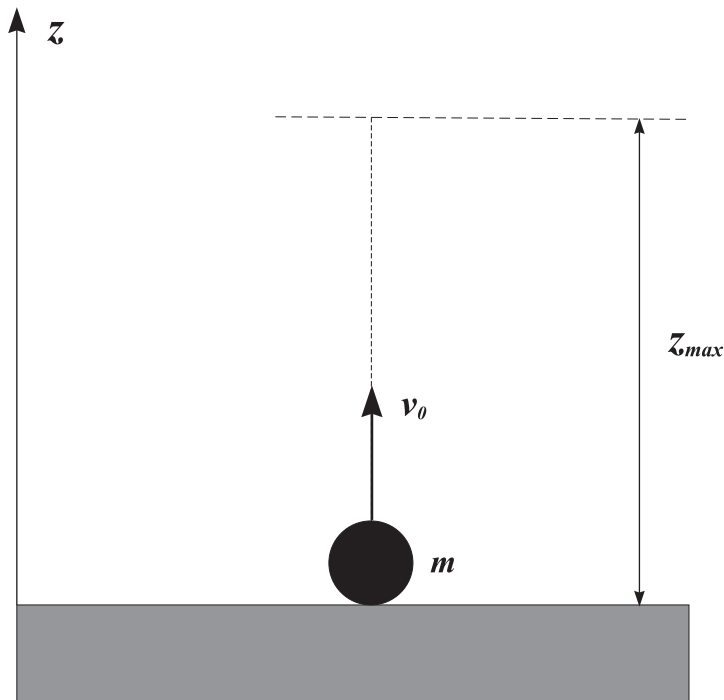


Figure 1.2. The main scaling characteristics of a mass m , launched with initial speed v_0 , are easily predicted by dimensional analysis.

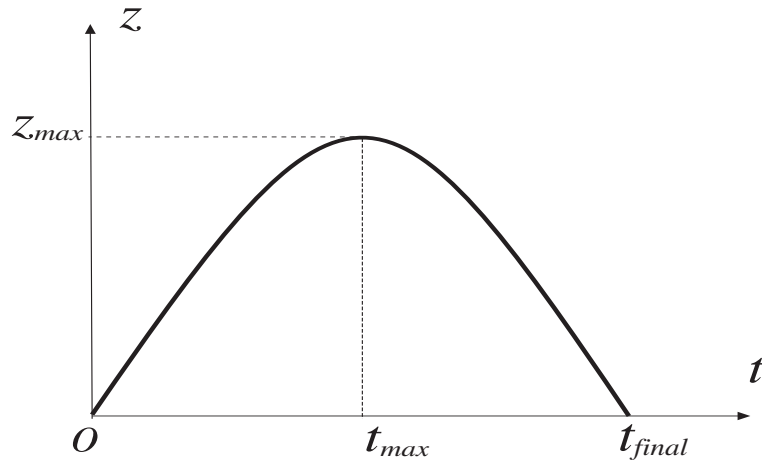


Figure 1.3. The height z of a mass, launched at speed v_0 , as a function of time t . It will reach a maximum height z_{max} at time t_{max} and reach the earth's surface again at time t_{final} .

Note that the mass m is not present in t^* and z^* , since the physical dimension M is not present in one of the other variables and parameters. The Buckingham theorem yields that its motion is described by a relation between z^* and t^* . This immediately leads to the conclusion that the motion of the projectile is independent of its mass. From experimental evidence we know that the relation between z and t is more or less as sketched in Fig. 1.3. The function $z(t)$ reaches a maximum z_{max} at t_{max} and vanishes at t_{final} . Since z^* and t^* are just scaled versions of z and t , z^* apparently can be written as an explicit function of t^* :

$$z^* = f^*(t^*). \tag{1.2}$$

The theorem does not specify any information about f^* but only ensures its existence and the insight that the form of f^* does not depend on any of the parameters m , v_0 , and g separately. The latter property thus also holds for the dimensionless quantities z_{max}^* , t_{max}^* , and t_{final}^* . These are just fixed numbers, as shown in Exercise 1.3a. Using the relations between dimensional and dimensionless quantities, we have that

$$z_{max} = \frac{v_0^2}{g} z_{max}^*, \quad t_{max} = \frac{v_0}{g} t_{max}^*, \quad t_{final} = \frac{v_0}{g} t_{final}^* .$$

This yields the insight that z_{max} scales with v_0^2 and both t_{max} and t_{final} with v_0 for fixed value of g . We denote this as

$$z_{max} \sim v_0^2, \quad t_{max} \sim v_0, \quad t_{final} \sim v_0 .$$

So, launching with a twice-as-large velocity leads to a four-times-larger maximal height of the projectile. In the same way we conclude that

$$z_{max} \sim \frac{1}{g}, \quad t_{max} \sim \frac{1}{g}, \quad t_{final} \sim \frac{1}{g}$$

for a fixed value of v_0 . So, catapulting on the moon, where g is (approximately six times) smaller than on the earth, enhances z_{\max} , t_{\max} , and t_{final} all by the same factor. \square

Exercise 1.3a.

Check these conclusions on catapulting by explicitly solving the equation of motion

$$m \frac{d^2 z}{dt^2} = -mg.$$

Show that f^ in (1.2) has the explicit form as given in Fig. 1.4. Calculate explicitly the values of z_{\max}^* , t_{\max}^* , and t_{final}^* . Note that this function cannot be found from dimensional analysis only.*

Example 1.3b. Swinging pendulum.

Consider the motion of a mathematical swing: this pendulum has mass m concentrated in a point at the end of a rigid rod of length ℓ . The motion is restricted to a vertical plane. See Fig. 1.5. The position of the swinging pendulum is completely specified by the angle φ with the vertical. This is the independent variable, and time t is the dependent variable. Parameters are mass m , rod length ℓ , gravitational acceleration g , and the initial position $\varphi_0 = \varphi(0)$. For convenience we take the initial velocity vanishing. So, $k + \ell = 6$, and since the three fundamental dimensions M , L , and T are involved, the system has three dimensionless quantities. Since φ and φ_0 are already dimensionless, they form an obvious choice. To find the third, we form the products

$$t^{r_1} \ell^{r_2} m^{r_3} g^{r_4}.$$

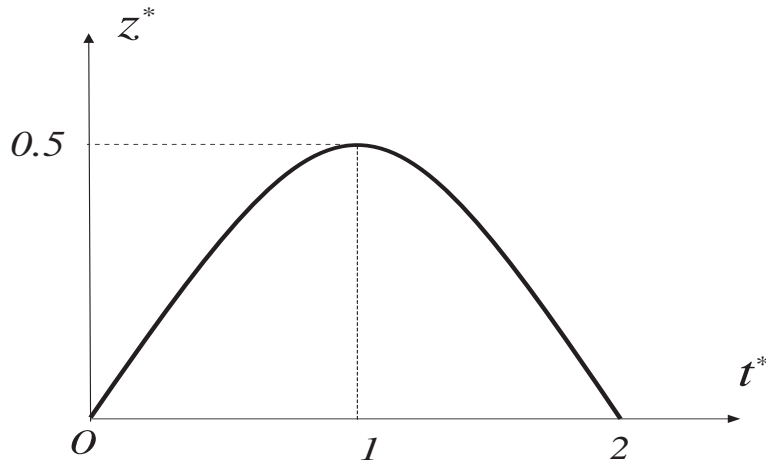


Figure 1.4. Explicit form of the dimensionless function f^* in (1.2). Note that this function is independent of the parameters m , g , and v_0 of the system. The dimensionless height z^* reaches a maximum value $z_{\max}^* = 1/2$ at time $t_{\max}^* = 1$ and hits the earth's surface again at time $t_{\text{final}}^* = 2$.

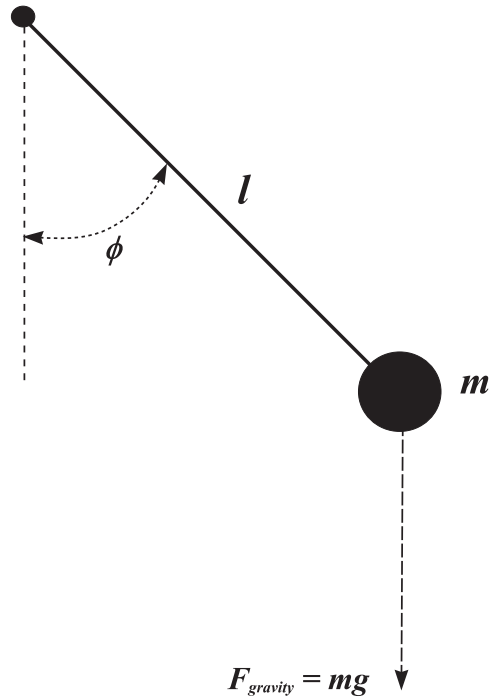


Figure 1.5. Swinging pendulum of mass m and length l . The motion is confined to a vertical plane, and the position of the pendulum can be indicated by the angle ϕ with the vertical.

The condition that this product must be dimensionless leads to the linear equations

$$\begin{aligned} r_1 - 2r_4 &= 0, \\ r_2 + r_4 &= 0, \\ r_3 &= 0. \end{aligned}$$

The choice $(r_1, r_2) = (1, 0)$ then yields

$$t^* = t \sqrt{\frac{g}{l}}.$$

Note that the mass m is not present in any of the dimensionless quantities φ , φ_0 , and t^* . This implies that pendulum motion is independent from m . The movement of the pendulum is given by some relation between φ , φ_0 , and t^* . With φ_0 constant and t^* monotonously increasing, we may write φ as an explicit function of t^* :

$$\varphi = f^*(t^*, \varphi_0).$$

This allows for a conclusion about the period of the system. One should realize that dimensional analysis as such does not reveal that φ is a periodic function of time. However, if

we take this for granted in view of the observations, we have that $f^*(t^* + \tau^*) = f^*(t^*)$, with τ^* the dimensionless period. Since $\tau^* = \tau\sqrt{g/\ell}$ and τ^* does not depend on any of the parameters, we find that τ scales with $\sqrt{\ell/g}$, so

$$\tau \sim \sqrt{\frac{\ell}{g}}. \quad \square$$

Exercise 1.3b.

- a. Give the dimensionless form of the exact pendulum equation

$$m\ell\ddot{\varphi} + mg \sin \varphi = 0.$$

- b. If $|\varphi| \ll 1$, the linearized pendulum equation

$$m\ell\ddot{\varphi} + mg\varphi = 0$$

is a good approximation. Give its dimensionless form.

- c. Write the solution of the equation under b and check that the period indeed scales with $\sqrt{\ell/g}$ as derived in Example 1.3b. Determine how the period is influenced if the length is doubled and also when the pendulum is placed on the moon.

Example 1.3c. Harmonic oscillator.

Here, we revisit the harmonic oscillator introduced in Example 1.2a. Setting the initial values at zero for convenience, the model equation

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \omega t$$

has the two variables u and t and the five parameters m, c, k, F_0 , and ω . So, $N = 7$ in this case. The fundamental dimensions involved are mass M , length L , and time T . Forming the products

$$u^{r_1} t^{r_2} m^{r_3} c^{r_4} k^{r_5} F_0^{r_6} \omega^{r_7}$$

and substituting the dimensions, we arrive at the products

$$L^{r_1} T^{r_2} M^{r_3} \left(\frac{M}{T}\right)^{r_4} \left(\frac{M}{T^2}\right)^{r_5} \left(\frac{ML}{T^2}\right)^{r_6} \left(\frac{1}{T}\right)^{r_7}.$$

Collecting powers of M, L , and T , we obtain the following three linear equations for the r_i :

$$\begin{aligned} r_1 + r_6 &= 0, \\ r_2 - r_4 - 2r_5 - 2r_6 - r_7 &= 0, \\ r_3 + r_4 + r_5 + r_6 &= 0. \end{aligned}$$

Here, we meet with three equations for seven unknowns, so four unknowns can be treated as free parameters. For example, we could take r_1, \dots, r_4 . The choices $(r_1, r_2, r_3, r_4) = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0),$ and $(0, 0, 0, 1)$, respectively, yield the dimensionless quantities

$$u^* = \frac{uk}{F_0}, \quad t^* = \omega t, \quad m^* = \frac{m\omega^2}{k}, \quad c^* = \frac{c\omega}{k}.$$

The dimensionless spring equation then reads as

$$m^* \ddot{u}^* + c^* \dot{u}^* + u^* = \sin t^*,$$

where the time derivative is taken with respect to t^* . □

Exercise 1.3c.

The approach used above for the driven spring system is based on the assumption $F_0 \neq 0$. Apply dimensional analysis to the case $F_0 = 0$ but with the initial position u_0 and initial velocity v_0 both nonvanishing.

Example 1.3d. Estimating the power of explosions.

Details of the strength of the first atomic bomb in 1945 were classified until the 1960s. However, the British physicist G.I. Taylor was able to give a very accurate estimate of the strength from dimensional analysis by using available film of the expansion of the mushroom shape of the explosion. His arguments proceed as follows (see, e.g., [31] and [3, 4]).

The basic appearance of the explosion is an expanding spherical fireball whose edge corresponds to a powerful shock wave, as sketched in Fig. 1.6. Let R be the radius of the shock wave. It will depend on E , the energy released by the explosion; t , the time elapsed since the explosion; ρ , the initial and ambient air density, and p , the initial and ambient

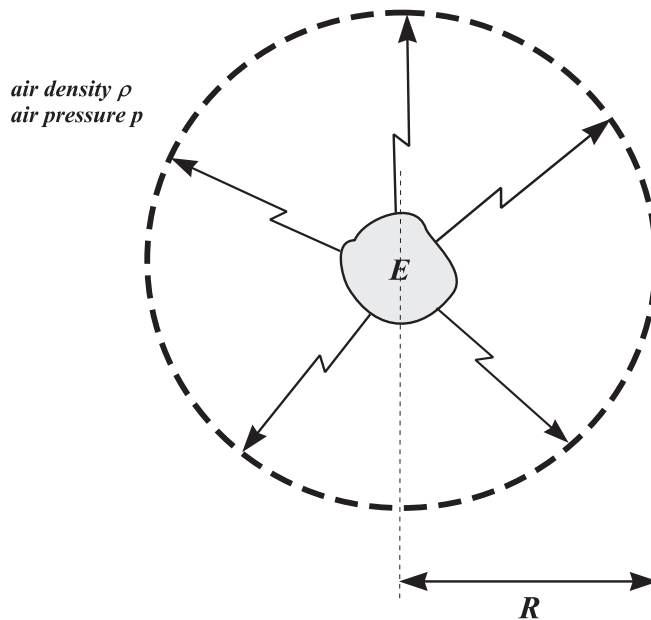


Figure 1.6. Sketch of a shock wave propagating from an explosion with energy E . Dimensional analysis shows that the energy E can be estimated from the propagation velocity of the front.

air pressure. In total we recognize five variables and parameters. Three dimensions are involved. Hence two dimensionless quantities can be found.

Exercise 1.3d.

The dimensions of the variables and parameters in Example 1.3c can be looked up in the tables in §1.2. Show that from requiring the products

$$R^{r_1} t^{r_2} E^{r_3} \rho^{r_4} p^{r_5}$$

to be dimensionless, the following dimensionless quantities can be derived:

$$q_1 = p \left(\frac{t^6}{E^2 \rho^3} \right)^{1/5}, \quad q_2 = \frac{R^5 \rho}{Et^2}.$$

The Buckingham theorem ensures that the motion of the shock front is governed by some relation between q_1 and q_2 . Since q_1 , which is essentially a scaled time, is monotonously increasing, we may write q_2 as an explicit function of q_1 : $q_2 = f^*(q_1)$. Thus R can be expressed as

$$R = \left(\frac{Et^2 f^*(q_1)}{\rho} \right)^{1/5}. \tag{1.3}$$

From this relation we conclude in the first instance that R depends on t both via the prefactor of f^* and via q_1 . This complicates the analysis. Taylor found a way out by first plotting measured values of R as a function of t in a double logarithmic plot. Taking logarithms of both sides of (1.3) we have

$$\log R = \frac{2}{5} \log t + \frac{1}{5} \log \left(\frac{E f^*(q_1)}{\rho} \right).$$

The data turn out to lie nearly on a straight line. This suggested that $f^*(q_1)$ hardly depends on time so that it can be replaced with its initial value $f^*(0)$. Then, two unknown parameters still figure in the model: E and $f^*(0)$. Taylor estimated the value of $f^*(0)$ from performing an experiment under well-defined conditions, for which E was known and $R(t)$ measured. This led to the conclusion that $f^*(0) \approx 1$. Every shock wave is thus fairly well described by the model equation

$$\log R = \frac{2}{5} \log t + \frac{1}{5} \log \left(\frac{E}{\rho} \right).$$

Since ρ is usually known, it is easy to estimate E from fitting this equation into measured $R(t)$ data. □

Example 1.3e. Estimating train speed from drop tracks.

Let us imagine we are traveling by train on a rainy day. Looking through the window we see rain drops attached to the glass following straight trajectories along the window downward. The angle of inclination of the drop paths appears to depend on the train speed. We wonder whether the speed can be estimated from this angle. To answer this question, the system

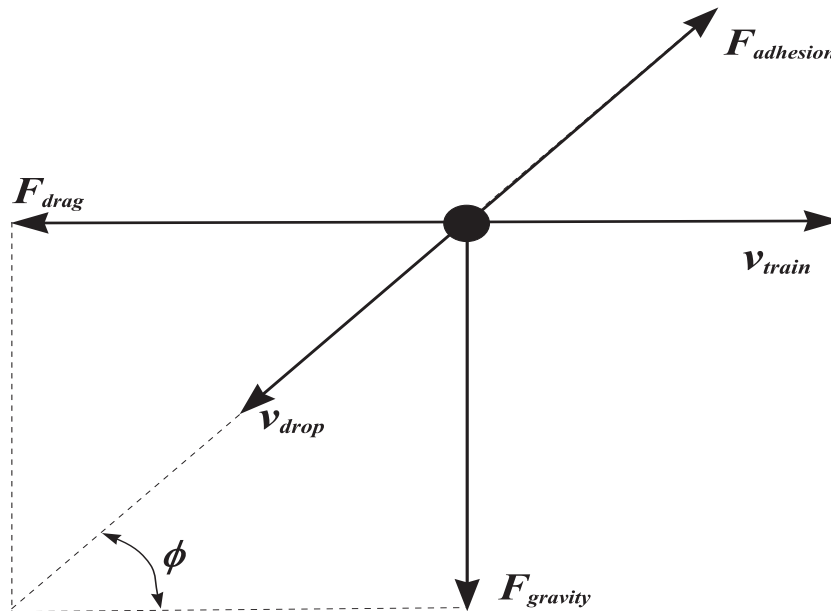


Figure 1.7. The path of a water droplet on a train window yields enough information to estimate the train speed, as follows from dimensional analysis.

of a drop sliding along a moving glass plate has to be modeled. See Fig. 1.7. Three forces act on the droplet: the gravitational force with strength F_g , the drag force with strength F_d due to friction between the drop and surrounding air, and the adhesion force with strength F_a between drop and glass. F_g is directed vertically, and its strength is equal to mg with m the drop mass. F_d is directed horizontally, but its strength is not known beforehand. F_a is a friction force and directed in the reverse direction of the drop speed. This force influences the speed of the drop but not its direction. This implies that the angle of inclination φ of the drop trajectory is determined by F_g and F_d . From Fig. 1.7 we conclude that

$$\tan \varphi = \frac{F_g}{F_d}.$$

To estimate the train speed v_{train} from this relation, we must know how F_d depends on v_{train} . We investigate how dimensional analysis can help us to discover this relation. The friction between the drop and the passing air will depend on drop diameter \mathcal{D} with $[\mathcal{D}] = L$, air density ρ with $[\rho] = M/L^3$, and air speed given by v_{train} with $[v_{\text{train}}] = L/T$. F_d is a force, so $[F_d] = ML/T^2$. The friction force exerted by a flow on an object moving through the flow is measured by the viscosity η with $[\eta] = M/LT$. It measures the internal friction. Common sense tells us that syrup has a larger viscosity than water. The value of η for a gas or fluid can be measured by dropping an object in the medium. Under influence of gravity it will initially accelerate. After some transient time its speed will become constant, since then the friction force comes into equilibrium with the gravitational force. This phenomenon is experienced by, e.g., parachutists. The viscosity can directly be deduced from

the equilibrium speed of a standardized object. In determining an expression for F_d the five quantities F_d , v_{train} , D , L , and η are involved. Since three physical dimensions play a role, the system has two dimensionless quantities.

Exercise 1.3e.

Check that we can choose for these quantities:

$$F_d^* = \frac{F_d}{\rho D^2 v_{\text{train}}^2}, \quad Re = \frac{\rho D v_{\text{train}}}{\eta}.$$

The dimensionless *Reynolds number* Re plays an important role in all flow problems. It measures the ratio of the convective and viscous forces and is named after Osborne Reynolds, a researcher in fluid mechanics. We expect that for a still unknown function f^* it will hold that

$$F_d^* = f^*(Re).$$

For the drag force we thus have

$$F_d = \rho D^2 v_{\text{train}}^2 f^*(Re).$$

From this we cannot deduce how F_d scales with v_{train} since Re also contains v_{train} . To answer this intricate question one has to determine the form of f^* from measurements. These data are given in Fig. 1.8. Note that this form is universal and holds for all flows, thanks to the dimensionless formulation. The conclusion from these data is that

$$f^*(Re) \sim \begin{cases} \frac{1}{Re} & \text{if } Re < 10, \\ 1 & \text{if } Re > 100. \end{cases}$$

The range $10 < Re < 100$ is a transition region. To find the order of magnitude of Re for a moving train, we substitute some data. For air we have $\rho \approx 1.3 \text{ kg/m}^3$ and $\eta \approx 1.5 \cdot 10^{-5} \text{ kg/(m.s)}$. The size of a droplet is $D \approx 5 \cdot 10^{-3} \text{ m}$. The velocity of the train varies from 0 to, say, 50 m/s. Substituting these numbers we find that $Re > 1000$ if $v_{\text{train}} > 2.5 \text{ m/s}$ (i.e., 10 km/h). This leads to the conclusion that for all relevant train speeds we have that

$$F_d = c_1 \rho D^2 v_{\text{train}}^2$$

for some constant c_1 which does not depend on ρ , D , and v_{train} . Eventually, we arrive at

$$v_{\text{train}} = \left(\frac{mg}{c_1 \rho D^2 \tan \varphi} \right)^{1/2}.$$

This expression can be reduced a bit by noting that the drop is approximately a half sphere, so $m = \frac{1}{12} \pi D^3$ if we take the density of water equal to unity. Since g and ρ hardly vary, we may write

$$v_{\text{train}} = c_2 \frac{\sqrt{D}}{\sqrt{\tan \varphi}}.$$

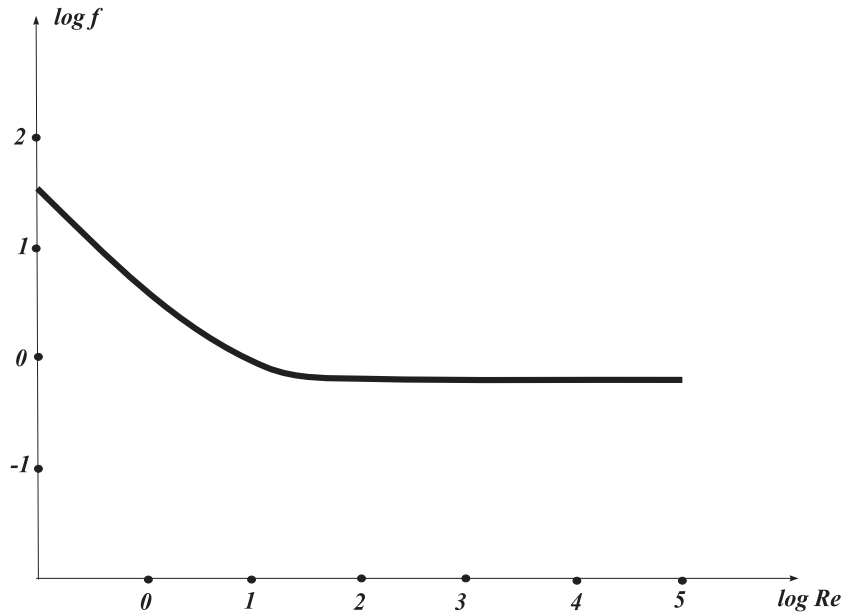


Figure 1.8. The drag force felt by an object in a flow as a function of the Reynolds number Re .

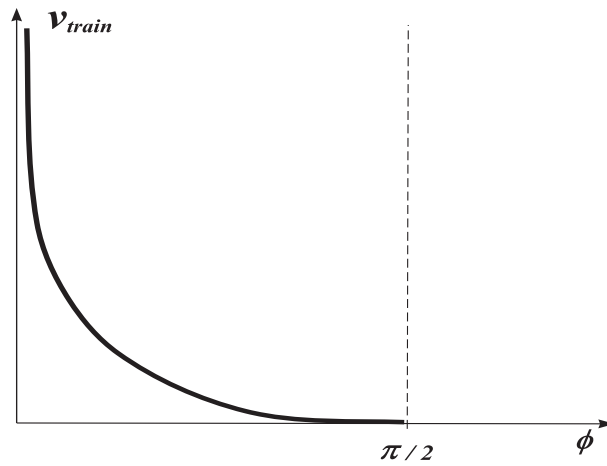


Figure 1.9. The dimensionless train velocity as a function of the angle ϕ indicated in Fig. 1.7.

The value of c_2 can be determined from one observation. This is the relation we were aiming at. In Fig. 1.9 we plot the dimensionless velocity $v_{\text{train}}^* \equiv v_{\text{train}} / (c_2 \sqrt{D})$ as a function of ϕ . Note that the accuracy with which ϕ must be measured becomes increasingly important if ϕ becomes smaller and smaller and thus the train speed becomes increasingly higher. \square

Example 1.3f. Ship modeling.

Let us model a ship of length ℓ sailing at constant speed v , as sketched in Fig. 1.10. The motion of the ship transfers energy from the ship to the water as a result of viscous friction. This energy is used partly to induce surface waves and partly to overcome the internal friction of the turbulent motion of the water. In view of these effects, the acceleration of gravity g , the density of water ρ , and the viscosity η each will play a role, with dimensions $[g] = L/T^2$, $[\rho] = M/L^3$, and $[\eta] = M/LT$, respectively. If we assume that the ship is streamlined such that its height and width are not of importance, the system has five variables and parameters. Because three dimensions are involved, the number of dimensionless quantities is two. We can choose for these quantities

$$Fr = \frac{v}{\sqrt{\ell g}}, \quad Re = \frac{\rho \ell v}{\eta}.$$

Fr is called the *Froude number* after William Froude, a famous ship builder. Re is the Reynolds number that we already met in Example 1.3.e. Because real-life experiments are difficult to conduct for these systems, it is very attractive to perform experiments on (physical) models in which all sizes are scaled down by a certain factor. The conclusions from these experiments are valid for the original system only if both systems are described by the same dimensionless (mathematical) model. So, Fr and Re have to remain constant upon scaling. In practice, the values of g , ρ , and η can hardly be adjusted. To keep Fr constant, $v/\sqrt{\ell}$ may not change, and to keep Re constant, $v\ell$ must be preserved. The experimenter must be aware that these requirements can never be managed in the same experiment. This implies that in one experiment with scaled ship models, only a restricted aspect of the real situation can be studied. □

Exercise 1.3f.

- a. Check that we can indeed choose

$$Fr = \frac{v}{\sqrt{\ell g}}, \quad Re = \frac{\rho \ell v}{\eta}$$

as dimensionless numbers.

- b. Why is it not possible to design the test facilities in such a way that both the Froude and the Reynolds number are preserved? Can you give an argument for why the

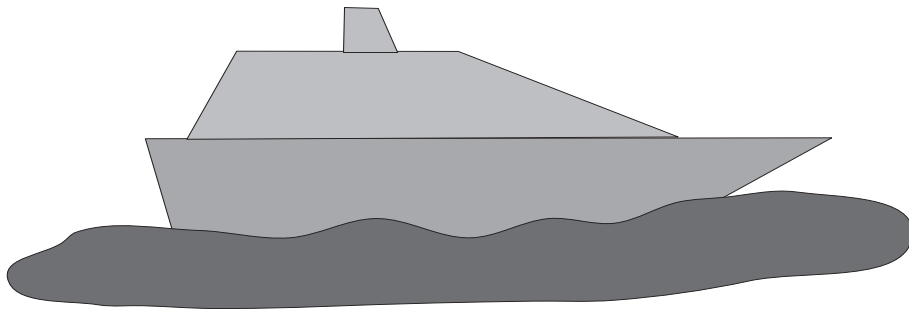


Figure 1.10. Study of scaled ship models is useful only if one realizes the consequences of dimensional analysis.

scaling of a ship is usually determined by the Froude number and not the Reynolds number? Will this be the same for testing an airplane in a (low-speed) wind tunnel?

- c. *To conduct experiments on a ship 100 m long that sails with a maximum speed of 35 km/hr, one uses in a laboratory a model of the ship. Towing in the laboratory is restricted to velocities of at most 7 km/hr. What is the smallest scale of the model that can be used?*

1.4 Scaling

The aim of *scaling* is to reduce the number of parameters in a given model. So, a prerequisite of the technique of scaling is knowledge of the equations governing the system. Scaling does not necessarily yield dimensionless quantities. We introduce the technique via examples.

Example 1.4a. The scaled harmonic oscillator.

We apply the scaling technique to the harmonic oscillator introduced in Example 1.2a. As pointed out in that example and in Example 1.3f, this system has the two variables u and t and the seven parameters $m, c, k, F_0, \omega, u_0, v_0$. Consider the scaling of variables

$$\eta = \frac{u}{a}, \quad \tau = \frac{t}{b}$$

with the scaling factors a and b still unspecified. Substitution of this transformation into equation of motion (1.1) leads to

$$\frac{ma}{b^2} \ddot{\eta} + \frac{ca}{b} \dot{\eta} + k a \eta = F_0 \sin(\omega \tau),$$

where the time derivative is now with respect to τ . Dividing all terms by the factor ma/b^2 and choosing $b = \sqrt{m/k}$ and $a = F_0/k$ we obtain the reduced equation

$$\ddot{\eta} + c' \dot{\eta} + \eta = \sin(\omega' \tau)$$

with the new parameters $c' = c/\sqrt{km}$ and $\omega' = \omega\sqrt{m/k}$. The initial values u_0 and v_0 have to be scaled accordingly to u'_0 and v'_0 . Note that this reduction reveals that the harmonic oscillator depends on only four parameters (including the initial parameters). To study the behavior of the solution as a function of the parameters, it suffices to vary only the friction coefficient c' and the angular frequency ω' (apart from the initial values). We remark that the scaling procedure does not lead to a unique choice for the scaling factors a and b . An alternative choice is $b = 1/\omega$ and $a = F_0/(\omega^2 m)$, which yields the equation

$$\ddot{\eta} + c' \dot{\eta} + k' \eta = \sin(\tau)$$

with $c' = c/(\omega m)$ and $k' = k/(\omega^2 m)$. So, for a complete analysis of the system it suffices to vary only these c' and k' (apart from the initial values). □

Exercise 1.4a.

Compare the methods and results of scaling and dimensional analysis when applied to the driven, linear spring in Examples 1.3 and 1.3b.

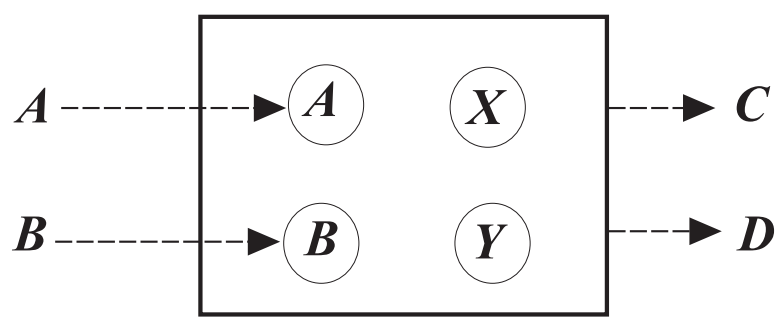
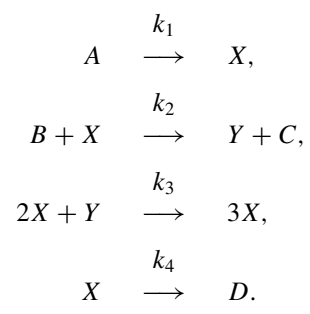


Figure 1.11. Sketch of the chemical reaction in Example 1.4b. The substances A and B are continuously supplied so that their concentrations are constant. The intermediate products X and Y are formed in reactions between A and B. C and D are the final products.

Example 1.4b. Chemical reaction.

Consider a hypothetical chemical reaction, the so-called Brusselator, with substances A, B, C, D, X, Y involved. The situation is sketched in Fig. 1.11. C and D are produced from A and B with X and Y as intermediates. The reaction has the following irreversible stages:



The capital letters denote reagents, while the constants k_i over the arrows indicate the reaction rates. It is assumed that A and B are excessively available so that the concentrations of A and B can be taken to be constant. We denote the concentrations of A, B, X, and Y by $a, b, x,$ and $y,$ respectively. The reaction equations for x and y are then

$$\begin{aligned}
 \dot{x} &= k_1 a - k_2 b x - k_4 x + k_3 x^2 y, \\
 \dot{y} &= k_2 b x - k_3 x^2 y.
 \end{aligned}
 \quad \square$$

Exercise 1.4b.

Use scaling of (x, y, t) to (η, ξ, τ) to show that these equations can be reduced to

$$\begin{aligned}
 \dot{\eta} &= \alpha - (\beta + 1)\eta + \eta^2 \xi, \\
 \dot{\xi} &= \beta \eta - \eta^2 \xi.
 \end{aligned}$$

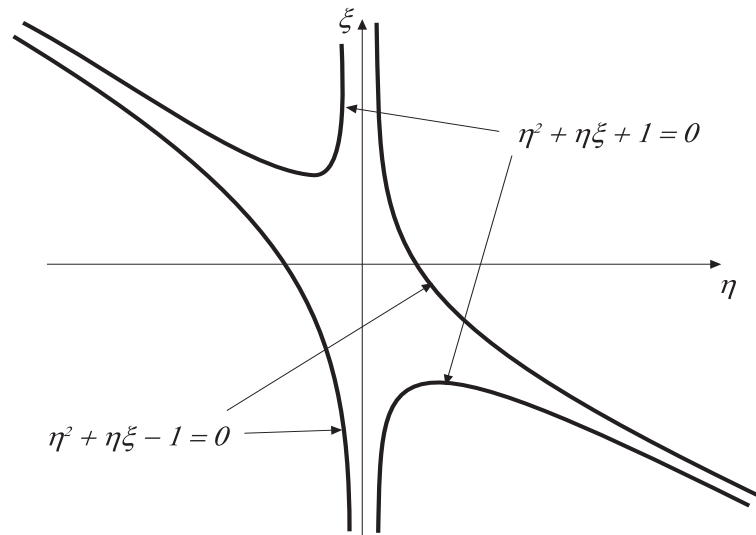


Figure 1.12. Solution sets of (1.5) after scaling.

Note that after this procedure the number of parameters is considerably reduced, since instead of the original parameters a, b, k_1, \dots, k_4 , the final set of equations contains only the two parameters α and β .

Exercise 1.4c.

Use the method of scaling to show that the equation

$$x^2 + axy + b = 0, \tag{1.4}$$

with variables x, y and parameters a, b , can be reduced to the equivalent equation

$$\eta^2 + \eta\xi + \text{sign}(b) = 0 \tag{1.5}$$

with variables η, ξ . This implies that the structure of the solutions of (1.4) can be caught without varying the parameters. It suffices to study (1.5) for the two cases $\text{sign}(b) = +1$ and -1 . These curves are plotted in Fig. 1.12.

1.5 Challenging problems

In this section we challenge the reader to apply the techniques of dimensional analysis and scaling to situations that are quite difficult to model. For application of dimensional analysis we need to know only the dimensions of the variables and parameters involved in the system. For scaling, one needs to start from the governing equations. In the present cases the reader must take these equations for granted; they will be derived in Chapter 2. Here, the focus is on the reduction of the models via scaling and not on the derivation of the models themselves.

1.5.1 The Prandtl–Blasius problem

The following description of flow above a flat plate is a simple model with which to study, for example, the following phenomena:

- The disturbance of the air by a thin airfoil of an airplane that cruises at constant altitude with constant speed U .
- The disturbance of a strong, uniform wind by vegetation, buildings, etc.
- The disturbance of water flowing over a rough surface.

In all these cases, the flow (air, water) near the surface experiences resistance from the presence of the airfoil, obstacles, etc. In an idealized way, we model the airfoil or the ground by a horizontal flat plate (standing still) with the air or water flowing over it, assuming that the flow is uniform before it reaches the plate. We choose coordinates such that the plate is in the half plane ($x \geq 0, -\infty < y < \infty, z = 0$). See Fig. 1.13. Since the y -coordinate is irrelevant, we shall omit it. In front of the plate edge, where $x \leq 0$, the flow is uniform with velocity U in the positive x -direction. The fluid has mass density ρ with $[\rho] = M L^{-3}$ and viscosity η with $[\eta] = M L^{-1} T^{-1}$. Close to the plate the friction between plate and flow decelerates the flow. This friction causes the flow velocity near the

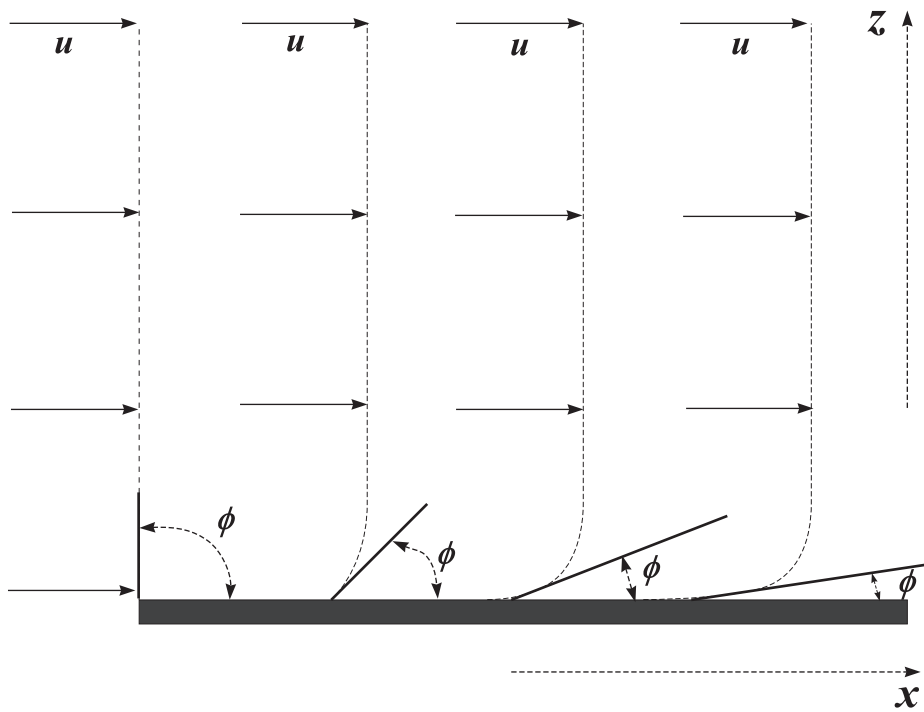


Figure 1.13. Time development of the velocity profile of air flow approaching and behind a trailing edge.

plate to drop. For increasing x , the boundary condition will approach the no-slip condition with the velocity at the plate exactly vanishing. For large x values, the velocity profile will become independent of x and approach a stationary profile, independent of x and thus t . We want to know how fast this convergence takes place as a function of time t and thus of position x . In Fig. 1.13 the velocity profiles above the plate are sketched for different x -positions. Let the velocity of the flow be denoted by (u, w) with u the velocity in the x -direction and w the velocity in the normal z -direction. The so-called shear rate is the variation of the horizontal velocity u in the normal direction. This is commonly denoted as $\dot{\gamma}$ (pronounced “gammadot”):

$$\dot{\gamma}(x, z) = \frac{\partial u}{\partial z}(x, z). \tag{1.6}$$

Its value at the plate is denoted as

$$\dot{\gamma}_0(x) = \dot{\gamma}(x, 0). \tag{1.7}$$

This quantity depends only on the distance x from the edge of the plate, and it is this dependence that we want to investigate. In Fig. 1.13 an angle $\varphi(x)$ is indicated. It is related to $\dot{\gamma}_0(x)$ via $\dot{\gamma}_0(x) = \tan \varphi(x)$. Far from the plate, where z is large, the flow is little influenced by the presence of the plate, and so there we may take $(u, w) = (U, 0)$.

In addition to the dependence on the distance x from the edge, the shear rate also will depend on the viscosity, the velocity U , and the density ρ . In the steady state we can generally write

$$\dot{\gamma}_0 = \dot{\gamma}_0(x; \eta, U, \rho).$$

In the following steps we want to find this relationship as precisely as possible. We first determine how far we can go with dimensional analysis in the first two exercises, whereas in the rest of this section we will apply scaling and use information from the governing equations.

- a. Determine two dimensionless variables from the set $\dot{\gamma}_0, x, \eta, U$, and ρ .
- b. Show that for some function f it holds that

$$\dot{\gamma}_0 = \frac{U^2 \rho}{\eta} f\left(\frac{U x \rho}{\eta}\right).$$

To apply scaling we need information about the governing equations. The actual equations of motion are given by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.8}$$

$$\rho \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \eta \frac{\partial^2 u}{\partial z^2}. \tag{1.9}$$

The first equation (vanishing of the divergence of the velocity field) expresses the incompressibility of the flow. The second equation expresses the balance between the convection force (at the left-hand side) and the viscous friction force (at the right-hand side).

We apply the scalings

$$\bar{u} = \frac{u}{U}, \quad \bar{w} = \frac{w}{W}, \quad \bar{x} = \frac{x}{X}, \quad \bar{z} = \frac{z}{Z}$$

with U given and X , Z , and W to be chosen later on.

c. Show that in (1.8) and (1.9) the number of parameters reduces if the following two conditions are satisfied:

$$\frac{X W}{Z U} = 1, \quad \frac{X \eta}{Z^2 U \rho} = 1. \tag{1.10}$$

In the rest of this section we assume these relations hold. Given Z , the second condition determines X , after which W follows from the first condition. So, of the three parameters W , X , and Z , only Z is still free to be chosen. Note that the scaled variables are not necessarily dimensionless.

d. Determine a scaling, i.e., choose Z , such that the scaled variables are dimensionless.

Now, we take Z again as a free parameter. The nonuniqueness of the scaling can then be exploited to find an explicit expression for the stress. Note that this can be done without solving the equations explicitly. The strategy is as follows. The scaled shear rate $\bar{\dot{\gamma}}_0$ at the plate is defined as

$$\bar{\dot{\gamma}}_0 := \frac{\partial \bar{u}}{\partial \bar{z}} = \frac{Z}{U} \dot{\gamma}_0. \tag{1.11}$$

Since $\dot{\gamma}_0$ depends only on x , it must hold for some function h that

$$\dot{\gamma}_0(x) = h(x). \tag{1.12}$$

e. Use (1.11), (1.12), and the fact that $x = X \bar{x}$ to find an expression for $\bar{\dot{\gamma}}_0(\bar{x})$ in terms of the function h . The resulting relation depends on Z . Since it must identically hold for any value of Z , we may draw a conclusion about the form of h . Show that for some positive constant c it must hold that

$$h(x) = \frac{c}{\sqrt{x}}.$$

We emphasize that this argument can be applied only because Z was kept arbitrary. If Z were fixed, e.g., to make all quantities dimensionless, this relation could not be derived.

f. Now, translating into the original, unscaled, quantities we show that

$$\dot{\gamma}_0(x; \eta, U, \rho) = c \sqrt{\frac{U^3 \rho}{\eta x}},$$

which is the relation we aimed at. Compare this with the result under b and determine the explicit form of the function f mentioned there.

1.5.2 Heat conduction in a bar

We consider heat conduction in a rod of length ℓ . The rod is assumed to be thermally isolated everywhere. Starting with a given temperature distribution $u(x, 0)$ over the rod, we are interested in the time evolution of the temperature profile $u(x, t)$. As will be explained in Example 2.3d in Chapter 2, the speed of the heat conduction along the rod is determined by the so-called thermal conductivity κ . The dimension of κ is L^2/T . As an initial profile we take a distribution which is everywhere vanishing except for a peak in the origin:

$$u(x, 0) = u_0 \delta(x), \quad (1.13)$$

where $\delta(x)$ is the so-called delta function. Its definition is given in §3.3.3.

a. Determine the dimension of the constant u_0 . To that end, integrate the initial condition over some interval including $x = 0$ and use the properties of the delta function.

The variables are thus u , x , and t , and the parameters are ℓ , κ , and u_0 .

b. Find three dimensionless quantities, choosing them such that they are just scalings of the variables u , x , and t .

Now we assume that the rod is infinitely long. This implies that the parameter ℓ is no longer relevant and that the number of dimensionless quantities equals two.

c. Show that these dimensionless quantities are appropriately chosen as $q_1 = x^2/(\kappa t)$ and $q_2 = u\sqrt{\kappa t}/u_0$.

In view of the expected behavior of the system we may write

$$q_2 = f^*(q_1) \quad (1.14)$$

for some function $f^*(q_1)$. The important conclusion from dimensional analysis is that in an infinitely long rod the temperature is a function not of position x separately but of the quotient x^2/t . Note that this does not hold for a rod of finite length ℓ . To find the explicit form of f^* , one needs the governing equation. The equation that describes the time evolution of the temperature in the bar follows from conservation of heat. It will be derived in Example 2.3d of Chapter 2. The resulting partial differential equation (PDE), the so-called heat diffusion equation, reads as

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}. \quad (1.15)$$

d. Show that the dimension of κ given above agrees with the dimension following from this equation.

e. Use (1.14) to rewrite the heat diffusion equation (1.15) in terms of q_1 , q_2 , and f^* , and show that this leads to the following ODE for $f^*(q_1)$:

$$4q_1 \frac{\partial^2 f^*}{\partial q_1^2} + (q_1 + 2) \frac{\partial f^*}{\partial q_1} + \frac{1}{2} f^* = 0 \tag{1.16}$$

for $0 < q_1 < \infty$.

f. Check that

$$f^*(q_1) = c e^{-\frac{1}{4}q_1}$$

is a solution of (1.16) for any constant c .

g. Which conservation principle can be invoked to determine c ? Check that the dimensional solution reads as

$$u(x, t) = \frac{u_0}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}.$$

Thus, at each time t the spatial temperature distribution is a Gaussian distribution in x , with mean 0 and standard deviation determined by κt . The temperature decays with increasing time, since the initial peak of heat is conducted along the rod. This is sketched in Fig. 1.14 for different values of t .

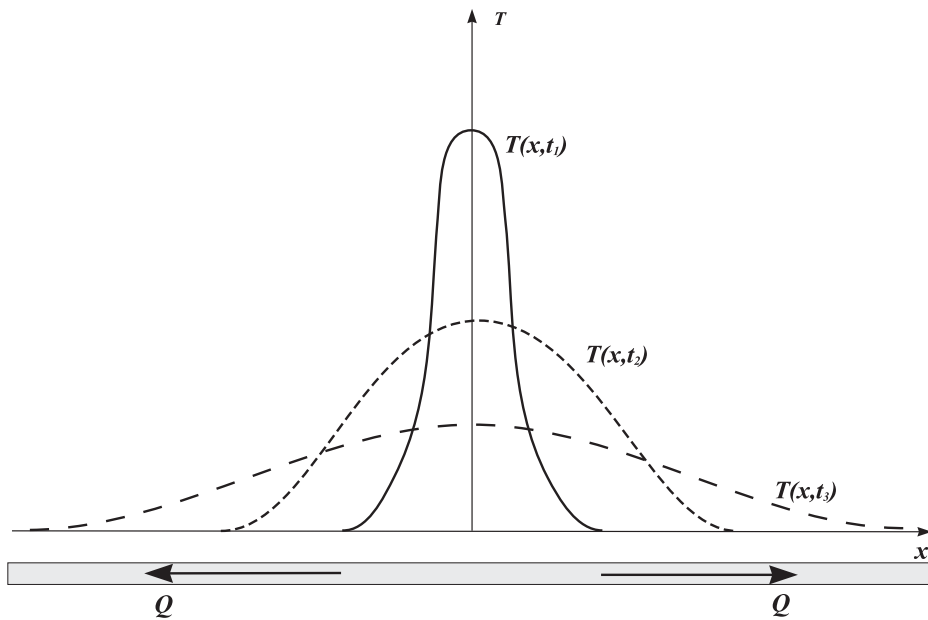


Figure 1.14. Temperature profiles for times $t_1 < t_2 < t_3$ in a long, thin rod.

1.5.3 Water waves

Consider a layer of fluid, for example, water, above a horizontal bottom. See Fig. 1.15. When the fluid is set into motion, by whatever cause, the fluid particles will start to move, interacting with each other, and are influenced by gravity. When the upper surface of the fluid is *free*, this surface will also be deformed by the particle motions. Looking only at the surface, the deformation is often of a characteristic type, like waves that are caused by throwing a stone in a pond. In principle, the particles below the surface determine the surface elevation. However, when the fluid is incompressible, and when so-called irrotational flow is considered, it is possible to describe approximately the elevation of the surface without reference to the internal motion. We will not *derive* the governing equations from first principles, but we will investigate a postulated description of the phenomenon via dimensional analysis and scaling. We make the following assumptions:

- The fluid is incompressible with uniform mass density, which we set equal to unity for convenience.
- No forces act from above on the free surface; for instance, effects of wind are neglected.
- In the horizontal plane, the motion is uniform in one direction; that is, we consider *plane waves*. This implies that if we take the x -axis, say, as the direction of propagation, the motion does not depend on the transverse horizontal direction.

We take the z -axis in the vertical direction, opposite to the direction of gravity with its origin at the undisturbed water surface. The distance between bottom and undisturbed water surface is H , and g denotes the acceleration of gravity. The modeling focuses on the elevation $u(x, t)$ of the surface with respect to the undisturbed level $z = 0$.

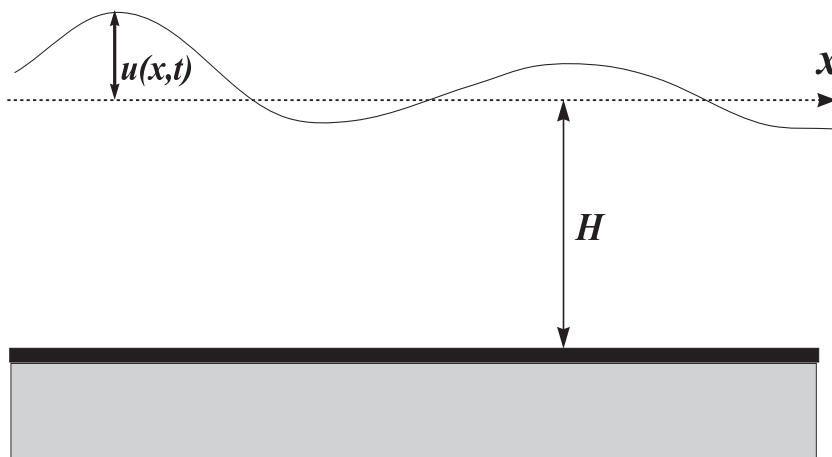


Figure 1.15. In the modeling of moving water waves one wants to find the elevation $u(x, t)$ of the surface with respect to the completely flat rest profile.

a. With the (constant) mass density removed from consideration, the problem has three variables (u, x, t), two parameters (g, H), and two dimensions (length and time). Find three dimensionless quantities.

A rough idea about wave propagation can be obtained by studying harmonic profiles, say, of the form

$$u(x, t) = a \cos\left(\frac{2\pi(x + Vt)}{\lambda}\right),$$

where a is the wave amplitude and λ is the wave length. This surface profile propagates at wave velocity V .

b. Take these three quantities into account, together with the parameters g and H , and show that the problem is described by a relation between the normalized amplitude, the wave length, and the dimensionless velocity:

$$f\left(\frac{a}{H}, \frac{\lambda}{H}, \frac{V}{\sqrt{gH}}\right) = 0.$$

c. From the above result, try to explain the observation—which can be made at any coast when one looks at the waves running into the shore—that waves approach the coast perpendicularly, even when the coastal boundary is irregular.

In 1895 Korteweg and de Vries [17, 24] published an equation for the elevation $u(x, t)$ that describes the surface elevation in a certain order of approximation. In the derivation of the equation it was assumed that the waves were “rather low” (small amplitude) and “rather long.” This equation is known as the *Korteweg–de Vries equation* (KdV equation). This equation became famous when it was found to have special mathematical properties. We do not discuss them here but instead focus on the original purpose of this model. It reads as

$$\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} - \frac{cH^2}{6} \frac{\partial^3 u}{\partial x^3} - \frac{3c}{2H} u \frac{\partial u}{\partial x}, \tag{1.17}$$

where the parameter $c = \sqrt{gH}$ has been introduced since it plays an important role in the physical phenomenon, as we shall see later. This equation shows the time evolution of the free surface: at a fixed position, the time derivative of the elevation depends (in a complicated way) on the spatial derivatives of the elevation: *temporal variations and spatial variations are coupled*, which is characteristic for a PDE. Understanding the coupling would mean that the meaning of each of the three terms in the right-hand side should be clear, which at this moment is not possible. The equation is rather difficult in the sense that it is not easy to find explicit solutions. In the following we will try to interpret the various terms in the right-hand side and, particularly, try to understand how the underlying modeling assumptions of long and low waves are shown in this equation.

The starting point is rather characteristic. We perform a scaling of the variables without specifying at this moment the scaling factors; these will be determined, or chosen, at a later instance.

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{\tau}, \quad \bar{u}(\bar{x}, \bar{t}) = \frac{u(x, t)}{a}.$$

Note that at this time we do not try to make the new variables $\bar{u}, \bar{x}, \bar{t}$ dimensionless. To simplify the notation and since confusion is unlikely, we drop the overhead bar, keeping in mind that the equations below are in terms of scaled quantities.

d. Show that the scaling leads to the following form of the equation:

$$\partial_t u + \alpha \partial_x u + \beta \partial_x^3 u + \gamma u \partial_x u = 0, \quad (1.18)$$

where

$$\alpha = \frac{\tau c}{L}, \quad \beta = \frac{\tau c H^2}{6L^3}, \quad \gamma = \frac{3\tau c a}{2HL}.$$

Here and often in the following, the spatial derivatives are indicated by $\partial_x u$ for $\frac{\partial u}{\partial x}$ and $\partial_t u$ for $\frac{\partial u}{\partial t}$, and similarly for higher-order derivatives. Observe that by the scaling we introduced three additional parameters (L , τ , and a), which together with H and g (or, equivalently, H and c) brings the total number of parameters to five. However, the equation shows that only specific combinations of these parameters (namely, α , β , and γ) play a role. We show how scaling can be used for various purposes.

To study the original KdV equation (1.17), it is sufficient to study (1.18). Therefore we would like to reduce the numbers of parameters in this equation as much as possible. This can be done by choosing the scaling coefficients in an appropriate way:

e. Determine scaling coefficients such that the KdV equation (1.18) gets the following, parameterless form:

$$\partial_t u + \partial_x u + \frac{1}{6} \partial_x^3 u + \frac{1}{6} u \partial_x u = 0. \quad (1.19)$$

This shows that by means of scaling, a lot of progress can be made. One needs only to study (1.19), and the results are directly applicable to various physical situations.

Scaling can be used for another purpose, as we shall show now. In particular we show how scaling arguments can give insight into the meaning and relative importance of the three terms in the right-hand side of (1.17) or, equivalently, of (1.18). As for comparisons in magnitude, it is necessary to make explicit that when dealing with the scaled variables, we assume these to be of unit order. So, in the following, x , t , and u and *all the derivatives of u* are considered to be of order unity. “Small” is then measured with respect to unity. Below, some limiting cases are considered.

Small amplitude waves

Since u is of order one, the value of a is a measure of the amplitude of the surface elevation: for large a the physical wave heights are large, whereas the limit $a \rightarrow 0$ means that waves of infinitesimal amplitude are considered. Since a appears only in the coefficient γ , we can simply take the limit $a \rightarrow 0$ by taking $\gamma \rightarrow 0$. The resulting equation is

$$\partial_t u + \alpha \partial_x u + \beta \partial_x^3 u = 0. \quad (1.20)$$

Note that this equation is linear. It can be solved by means of Fourier transformation techniques introduced in Chapter 3. This observation explains that the nonlinear term in the KdV equation describes effects that are due to the finiteness of the wave heights.

Long and short waves

The value of the parameter L determines the length of the physical spatial interval in which changes take place: for small L this interval is small, whereas for large L changes occur over a large physical interval. Small and large values of L correspond to short and long waves, respectively.

g. If we apply the transformation $X := Lx$, a function $f(x)$ transforms into a function $g(x)$, say, via

$$f(x) = g(X) = g(Lx).$$

Relate the derivatives of f to those of g and see the effect of the value of L .

All three parameters α , β , and γ in (1.18) contain the parameter L , and therefore the effect of taking a limiting value of L is not immediately clear. Thus, we argue as follows. The appearance of L in α can easily be scaled away by taking $\tau = L/c$. Then

$$\alpha = 1, \quad \beta = \frac{1}{6} \left(\frac{H}{L} \right)^2, \quad \gamma = \frac{3}{2} \left(\frac{a}{H} \right). \tag{1.21}$$

h. Show that if L is given the dimension of length, τ will have the dimension of time, and show that the variables x , t , and u and the parameters α , β , γ are dimensionless.

i. Observe that now L appears only in the coefficient β . Keeping all other parameters fixed, look at the limit for long waves, and explain that the third order spatial derivative in the KdV equation describes effects that are due to the length of the waves under consideration; the longer the waves, the less this term contributes. Find the equation obtained in the limit for infinite long waves of finite amplitude.

Long waves with small amplitudes

j. Consider the limit of infinitesimally small, infinitely long, waves. Show that by taking the limits $a \rightarrow 0$, $L \rightarrow \infty$ in (1.18), the equation reduces to

$$\partial_t u + \partial_x u = 0. \tag{1.22}$$

The solutions of this equation are easily found. Show that for arbitrary f it holds that

$$u(x, t) = f(x - t).$$

This solution represents that the profile f is translated into the direction of the positive x -axis, undisturbed in shape and at unit velocity. For this reason, this equation is called the translation equation.

k. Show, by realizing that x and t are scaled variables here, that infinitesimal waves of long wave length propagate at speed $c = \sqrt{gH}$.

Now, let us return to (1.18) with the scaling given by (1.21):

$$\partial_t u + \partial_x u + \frac{H^2}{6L^2} \partial_x^3 u + \frac{3a}{2H} u \partial_x u = 0. \quad (1.23)$$

In the reasoning of Korteweg and de Vries, the last two terms are improvements of the translation equation in the sense that (some) effects of the nonlinearity and dependence on wavelength are taken into account. Being corrections, the coefficients should be small compared to unity and so of order $\varepsilon \ll 1$. The KdV equation takes both effects into consideration. Let us assume that both effects are equally important, say, of the same order ε .

l. Show that this means that the KdV equation describes waves of amplitude a and wavelength L such that

$$\frac{a}{H} = \mathcal{O}(\varepsilon), \quad \frac{H}{L} = \mathcal{O}(\sqrt{\varepsilon}).$$

Note that the quotients a/H and L/H are dimensionless and that all length scales that are relevant for measuring the two effects depend on the (only!) physical length H that is present in the model. Waves for which amplitude and wavelength are related in this way are said to satisfy the Boussinesq assumption.

m. For experiments in a towing tank in hydrodynamic laboratories, wave heights (i.e., twice the amplitude) up to 10% of the depth are considered, for instance, waves of amplitude 0.3 m on a depth of 6 m. Determine the order of the wavelength of waves that satisfy the Boussinesq assumption.