An Elementary Version of the Landesman–Lazer Theorem

Problem 00-001, *by* Philip Korman (University of Cincinnati, Cincinnati, OH 45221-0025 (kormanp@math.uc.edu)).

Consider the Neumann problem (for $u = u(x)$)

$$u'' + f(u) - g(x) = 0 \quad \text{on } (0,1), \quad u'(0) = u'(1) = 0. \tag{1}$$

Assume that $f(u)$ is continuous for all $u \in \mathbb{R}$ and the limits (finite or infinite) $f(-\infty) = \lim_{u\to-\infty} f(u)$ and $f(\infty) = \lim_{u\to\infty} f(u)$ exist, and

$$f(\infty) < f(u) < f(-\infty) \quad \text{for all } u \in \mathbb{R}. \tag{2}$$

Assume that $g(x) \in C(0,1)$.

Show that problem (1) has a classical solution if and only if

$$f(\infty) < \int_0^1 g(x)\,dx < f(-\infty). \tag{3}$$

Discussion. The direction of the inequalities in (2) is the opposite of the usual ones. The problem is then easier if one considers the Neumann boundary condition. (The original paper of Landesman and Lazer [1] considered the Dirichlet boundary conditions.)

REFERENCE

[1] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech., 19 (1969/1970), pp. 609–622.

Status. The proposer has an elementary solution. Additional solutions are invited.