A Family of Infinite Series

Problem 01-001, *by* Jonathan M. Borwein (CECM, Simon Fraser University, Burnaby, BC, Canada).

This proposal is motivated by a recent problem in the American Mathematical Monthly [2] proposed by Donald E. Knuth. Knuth's problem is to evaluate

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!\,e^k}-\frac{1}{\sqrt{2\pi k}}\right).$$

While the series

$$S := \sum_{k=0}^{\infty}\frac{k^k}{k!\,e^k}$$

diverges, hence Knuth's correction factor, the series

$$S_N := \sum_{k=0}^{\infty}\frac{k^k}{(k+N)!\,e^k}$$

converges for all positive integers N. It occurred to me to ask whether this sum had a nice closed form. For $N = 1$, the sum is $1.71828182845904523 54\ldots$. This is clearly $e-1$. Emboldened by this I used the integer relation methods. Algorithms such as PSLQ and LLL attempt to determine if n floating point numbers are dependent over the rationals (see the interface at www.cecm.sfu.ca/projects/IntegerRelations/). Such techniques are described at length in a forthcoming book [1].

I soon discovered that S_N was a rational polynomial in e of degree N for the first 10 cases. It remained to prove the following:

> **Proposed Problem.** *Prove that $S_N = P_N(e)$, where P_N is a rational polynomial in e of degree N, and determine the coefficients of P_N.*

REFERENCES

[1] D.H. Bailey, J.M. Borwein, and K. Devlin, *The Experimental Mathematician: A Computational Guide to the Mathematical Unknown*, A.K. Peters, Ltd., Natick, MA, to appear.

[2] Donald E. Knuth, *Problem* 10832, Amer. Math. Monthly, 107 (2000), p. 863.

Status. This proposer has solved this problem. Other solutions are invited.