Solution of Problem 01-004 *by* A.M. Dawes (University of Western Ontario, London, ON, Canada).

Solution of (a). Consider the diagonal $m+n=k$, where $k \geq 2$. On this diagonal, each $d_{m,n}$ can be written as $a_{m,n}/b_{m,n}$ where $a_{m,n}$ is a positive integer and $b_{m,n} = 2^{m+n-2} = 2^{k-2}$.

This form for the denominator $b_{m,n}$ follows immediately from the initial conditions and the recurrence; we will show by induction on k that, for the numerator,

$$a_{i,k-i} = \sum_{j=0}^{i-1} \binom{k-1}{j}, \quad 1 \leq i \leq k-1.$$

Basis ($m+n=2$): $d_{1,1} = 1 = \sum_{j=0}^{0} \binom{1}{j} \big/ 2^{2-2}$.

Induction step ($m+n=k+1$): for $i=1$, we have $a_{1,k} = 1 = \sum_{j=0}^{0} \binom{k}{j}$; for $i=k$, we find $a_{k,1} = 2^k - 1 = \sum_{j=0}^{k-1} \binom{k}{j}$. For $1 < i < k$, we use the induction hypothesis on the diagonal $m+n=k$ and the recurrence

$$\begin{aligned} a_{i,k+1-i} &= a_{i-1,k+1-i} + a_{i,k+1-i-1} \\ &= \sum_{j=0}^{i-2} \binom{k-1}{j} + \sum_{j=0}^{i-1} \binom{k-1}{j} \\ &= \sum_{j=0}^{i-1} \binom{k}{j} \quad \text{as required.} \end{aligned}$$

Therefore $d_{m,n} = \sum_{j=0}^{m-1} \binom{m+n-1}{j} \big/ 2^{m+n-2}$.

Solution of (b). In particular,

$$a_{m,m} = \sum_{j=0}^{m-1} \binom{2m-1}{j} = \frac{1}{2} \sum_{j=0}^{2m-1} \binom{2m-1}{j} = \frac{1}{2} \cdot 2^{2m-1} = 2^{2m-2} = b_{m,m}.$$

Therefore $d_{m,m} = 1$.

Solution of (c). Consider the partial sums of the rows, i.e., let $e_{m,n} = \sum_{j=1}^{n} d_{m,j}$ for $n \geq 1$.

Claim. These partial sums satisfy the recurrence

$$e_{m,n} = 1 + \frac{e_{m-1,n} + e_{m,n-1}}{2} \quad \text{for } m,n > 1$$

with initial conditions

$$e_{1,n} = \sum_{j=1}^{n} \frac{1}{2^{j-1}} = 2 - \frac{1}{2^{n-1}},$$

$$e_{m,1} = d_{m,1} = \frac{2^m - 1}{2^{m-1}} = 2 - \frac{1}{2^{m-1}}.$$

Proof of claim. The initial conditions are immediate; and for $m, n > 1$ we have

$$\begin{aligned} e_{m,n} &= \sum_{j=1}^{n} d_{m,j} = d_{m,1} + \sum_{j=2}^{n} d_{m,j} \\ &= 2 - \frac{1}{2^{m-1}} + \frac{1}{2}\sum_{j=2}^{n} d_{m-1,j} + \frac{1}{2}\sum_{j=2}^{n} d_{m,j-1} \\ &= 2 - \frac{1}{2^{m-1}} + \frac{e_{m-1,n}}{2} - \frac{d_{m-1,1}}{2} + \frac{e_{m,n-1}}{2} \\ &= 1 + \frac{e_{m-1,n} + e_{m,n-1}}{2}. \end{aligned}$$

Assuming convergence, we may now find the sums. Let l_m denote the limit of the partial sums of row m. Note first that the sum of the first row is 2, i.e., $l_1 = \sum_{j=1}^{\infty} d_{1,j} = 2$. Then for row $m+1$ we find $l_{m+1} = 1 + (l_m + l_{m+1})/2$, and it follows that $l_m = 2m$ for $m \geq 1$.

It remains to prove convergence; it suffices to show by induction on m that in each row the partial sums are bounded by $2m$. This is already done for row 1. For row $m+1$, proceed by induction on n:

$$e_{m+1,n} = 1 + \frac{e_{m,n} + e_{m+1,n-1}}{2}.$$

By the induction hypothesis on m, $e_{m,n} \leq 2m$; by the induction hypothesis on n, $e_{m+1,n-1} \leq 2(m+1)$; therefore $e_{m+1,n} \leq 1 + (m + m + 1) = 2(m+1)$.

Editorial note. An alternative solution is provided by the method of generating functions. It is convenient to extend the array $[a_{m,n}] = [2^{m+n-2} d_{m,n}]$ by setting

$$a_{0,n} = 0, \quad n \geq 0 \qquad \text{and} \qquad a_{m,0} = 2^{m-1}, \quad m \geq 1.$$

With this choice, the recurrence $a_{m,n} = a_{m-1,n} + a_{m,n-1}$ holds for all $m, n \geq 1$. Let

$$F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}\, x^m y^n.$$

Note that

$$\sum_{m=1}^{\infty} a_{m,0}\, x^m y = \sum_{m=1}^{\infty} 2^{m-1}\, x^m y = \frac{xy}{1-2x}.$$

Thus

$$\begin{aligned} F(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{m-1,n}+a_{m,n-1})x^m y^n \\ &= xF(x,y) + \frac{xy}{1-2x} + yF(x,y), \end{aligned}$$

so $F(x,y) = xy(1-2x)^{-1}(1-x-y)^{-1}$. For all $m,n \geq 1$,

$$\begin{aligned} a_{m,n} &= [x^m y^n]\ \frac{xy}{(1-2x)(1-x-y)} \\ &= [x^m y^n]\ \frac{xy}{(1-2x)(1-x)} \sum_{p=0}^{\infty}\left(\frac{y}{1-x}\right)^p \\ &= [x^m]\ \frac{x}{(1-2x)(1-x)^n} \\ &= [x^m]\ \frac{x}{(1-x)^{n+1}} \sum_{q=0}^{\infty}\left(\frac{x}{1-x}\right)^q \\ &= \sum_{q=0}^{m-1}\binom{m+n-1}{m-1-q} = \sum_{k=0}^{m-1}\binom{m+n-1}{k}. \end{aligned}$$

The last line uses the familiar fact that $[z^k](1-z)^{-a} = \binom{k+a-1}{k}$. This gives the desired result for part (a):

$$d_{m,n} = \frac{1}{2^{m+n-2}}\sum_{k=0}^{m-1}\binom{m+n-1}{k}.$$

For part (b), one can use the residue calculus to extract the diagonal terms of the generating function. If $|z|$ is sufficiently small and $C = \{w : |w| = r\}$ where $r < \frac{1}{2}$, the diagonal part of the generating function is

$$\sum_{m=1}^{\infty} a_{m,m}\, z^m = \frac{1}{2\pi i}\int_C \frac{F(w,z/w)}{w}\, dw = \frac{z}{2\pi i}\int_C \frac{dw}{(1-2w)(w-w^2-z)}.$$

The only contribution comes from the pole at $w = (1-\sqrt{1-4z})/2$, and straightforward calculation gives

$$\sum_{m=1}^{\infty} a_{m,m}\, z^m = \frac{z}{1-4z}.$$

Hence $a_{m,m} = 4^{m-1}$ and $d_{m,m} = 1$. Part (c) is brief. The generating function for $d_{m,n}$ is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{m,n}\, x^m y^n = 4F(\tfrac{x}{2}, \tfrac{y}{2}) = \frac{xy}{(1-x)(1-(x+y)/2)} := G(x,y).$$

Thus

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} d_{m,n}\right) x^m = G(x,1) = \frac{2x}{(1-x)^2},$$

and one finds

$$\sum_{n=1}^{\infty} d_{m,n} = [x^m]\frac{2x}{(1-x)^2} = 2m, \qquad m \geq 1.$$