Alternative Solution of a Double Recurrence

Solution of Problem 01-004 *by* T. R. Watts (Wool, Wareham, Dorset, England).

Solution of (a). To get a feel for the problem we first calculate the first few numbers $d_{m,n}$ and represent them in the matrix (m is the number of a row and n the number of a column):

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} \\ \frac{3}{2} & \frac{4}{4} & \frac{5}{8} & \frac{6}{16} & \frac{7}{32} & \frac{8}{64} & \frac{9}{128} \\ \frac{7}{4} & \frac{11}{8} & \frac{16}{16} & \frac{22}{32} & \frac{29}{64} & \frac{37}{128} & \frac{46}{256} \\ \frac{15}{8} & \frac{26}{16} & \frac{42}{32} & \frac{64}{64} & \frac{93}{128} & \frac{130}{256} & \frac{176}{512} \\ \frac{31}{16} & \frac{57}{32} & \frac{99}{64} & \frac{163}{128} & \frac{256}{256} & \frac{386}{512} & \frac{562}{1024} \\ \frac{63}{32} & \frac{120}{64} & \frac{219}{128} & \frac{382}{256} & \frac{638}{512} & \frac{1024}{1024} & \frac{1586}{2048} \\ \frac{127}{64} & \frac{247}{128} & \frac{466}{256} & \frac{848}{512} & \frac{1486}{1024} & \frac{2510}{2048} & \frac{4096}{4096} \end{pmatrix}$$

The fractions are not reduced to their lowest terms in order that patterns may be spotted more easily. Motivated by the numerical calculations we define the numbers $e_{m,n}$ by the equation

$$e_{m,n} = 2^{m+n-2} d_{m,n}.$$

It follows easily from the recurrence relation for $d_{m,n}$ that $e_{m,n}$ satisfies the recurrence

$$e_{m,n} = e_{m-1,n} + e_{m,n-1}$$

for $m, n > 1$. The initial conditions for $d_{m,1}$ and $d_{1,n}$ imply that

$$e_{1,n} = 1 \quad \text{and} \quad e_{m,1} = 2^m - 1$$

for all m and n. Then, for $m > 1$, we obtain

$$\begin{aligned} e_{m,n} &= e_{m,1} + \sum_{j=2}^{n} (e_{m,j} - e_{m,j-1}) \\ &= e_{m,1} - e_{m-1,1} + \sum_{j=1}^{n} e_{m-1,j} \\ &= 2^{m-1} + \sum_{j=1}^{n} e_{m-1,j}. \end{aligned}$$

This formula may be used to calculate $e_{m,n}$ (and consequently $d_{m,n}$) recursively from $e_{m-1,n}$. So

$$e_{2,n} = 2 + \sum_{j=1}^{n} e_{1,j} = n + 2, \quad \text{and therefore} \quad d_{2,n} = \frac{n+2}{2^n},$$

and

$$e_{3,n} = 2^2 + \sum_{j=1}^{n} e_{2,j} = 4 + \sum_{j=1}^{n} (j+2) = \frac{n^2 + 5n + 8}{2},$$

and therefore

$$d_{3,n} = \frac{n^2 + 5n + 8}{2^{n+2}}.$$

In order to derive similar expressions for $d_{m,n}$ for larger values of m, it is necessary to evaluate the sums

$$\sum_{j=1}^{n} j^k$$

for $k = 0, \ldots, m-2$. It is well known that these sums can be expressed as polynomials of degree $k+1$ in the variable n, where the coefficients of powers of n are given in terms of Bernoulli numbers (rational numbers that may be computed recursively).

We have shown that

$$d_{m,n} = \frac{\text{polynomial in } n}{2^{m+n-2}},$$

where the coefficients of the polynomial in the numerator depend on m and the Bernoulli numbers and the polynomial for row m can be determined from the polynomial for row $m-1$ by a simple recursion.

Solution of (b). We shall show that $d_{m,n} + d_{n,m} = 2$ for all m and n, which immediately implies that $d_{m,m} = 1$ for all m. Defining the numbers $f_{m,n}$ by the equation

$$f_{m,n} = d_{m,n} + d_{n,m},$$

for all m and n, it is easily shown that $f_{m,n}$ satisfies the same recurrence as $d_{m,n}$ and from the initial conditions defining $d_{1,n}$ and $d_{m,1}$ it follows that

$$f_{1,n} = f_{m,1} = 2$$

for all positive integers m and n. The identity $f_{m,n} = 2$ is then immediate (and may be formally proved by induction).

Solution of (c). Let the sum of row m be S_m, so that

$$S_m = \sum_{j=1}^{\infty} d_{m,j} = \lim_{N\to\infty} S_{m,N}, \quad \text{where} \quad S_{m,N} = \sum_{j=1}^{N} d_{m,j}.$$

When $m, N > 1$, it follows from the recurrence relation satisfied by $d_{m,n}$ that

$$\begin{aligned} 2S_{m,N} &= 2d_{m,1} + \sum_{j=2}^{N} (d_{m-1,j} + d_{m,j-1}) \\ &= 2d_{m,1} + S_{m-1,N} - d_{m-1,1} + S_{m,N} - d_{m,N}, \end{aligned}$$

which simplifies to

$$S_{m,N} = S_{m-1,N} + 2 - d_{m,N}.$$

In the answer to part (a) it was shown that

$$d_{m,N} = \frac{\text{polynomial in } N}{2^{m+N-2}}$$

for any fixed m. This implies that $d_{m,N} \to 0$ as $N \to \infty$. It then follows that the infinite series S_m converges if the infinite series S_{m-1} converges. But S_1 is a convergent geometric series with sum

$$S_1 = \sum_{j=1}^{\infty} d_{1,j} = \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 2.$$

Therefore, by induction, S_m is convergent for all $m > 1$ and

$$S_m = S_{m-1} + 2,$$

so that $S_m = 2m$ for all positive m.

Editorial note. This solution is of special interest since it shows that the correct results for parts (b) and (c) can be found without first having the explicit form for $d_{m,n}$ given by the proposer.