Rolewicz's Problem

Problem 01-005, *by* Bogdan Choczewski (Faculty of Applied Mathematics, University of Mining and Metallurgy (AGH), Krakow, Poland), Roland Girgensohn (Institute of Biomathematics and Biometry, GSF-Forschungszentrum, Neuherberg, Germany), *and* Zygfryd Kominek (Institute of Mathematics, Silesian University, Katowice, Poland).

The problem is motivated by the study of Rolewicz [3] (cf. also Pallaschke–Rolewicz [1]) of Fréchet $\mathbf{\Phi}$-differentiability of real-valued mappings of a metric space (X, d).

Let $\mathbf{\Phi}$ be a family of real-valued functions defined on X. A function $F : X \to \mathbb{R}$, lower semicontinuous on X, is said to be $\mathbf{\Phi}$-*convex* if there exists a $\mathbf{\Phi}_0 \subset \mathbf{\Phi}$ such that

$$F(x) = \sup \{\phi(x) : \ \phi \in \mathbf{\Phi}_0, \ \phi \le F\}, \qquad x \in X.$$

A function $\phi_0 \in \mathbf{\Phi}$ is called the $\mathbf{\Phi}$-*subgradient* of the function F at $x_0 \in X$ if

$$F(x) - F(x_0) \ge \phi_0(x) - \phi_0(x_0) \ , \qquad x \in X.$$

Let $\alpha : [0, +\infty) \to [0, +\infty)$ be a continuous function such that $\alpha(0) = 0$ and $\alpha(t) > 0$ for $t > 0$. Let $F : X \to \mathbb{R}$ be a $\mathbf{\Phi}$-convex function. The function $\varphi_0 \in \mathbf{\Phi}$ such that

$$F(x) - F(x_0) \ge \varphi_0(x) - \varphi_0(x_0) + \alpha(d(x, x_0)), \qquad x \in X, \tag{1}$$

is called a *strong* $\mathbf{\Phi}$-*subgradient* of the function F at the point x_0 *with the modulus* $\alpha(\cdot)$ and the set of all of them is said to be the *strong* $\mathbf{\Phi}$-*subdifferential with modulus* $\alpha(\cdot)$ *of* F *at* x_0.

Having denoted

$$\mathbf{\Phi}^\alpha =: \{\varphi + \alpha \circ d(\cdot, x_1), \ \varphi \in \mathbf{\Phi}, \ x_1 \in X\},$$

we observe that if φ is a strong $\mathbf{\Phi}$-subgradient of a function f at a point x_0 with the modulus $\alpha(\cdot)$, then $\phi := \varphi + \alpha \circ d(\cdot, x_0)$ is a $\mathbf{\Phi}^\alpha$-subgradient of f at x_0. However, the converse statement is no longer true; cf. Rolewicz [3] . Therefore, conditions are wanted for the two subdifferentials to be equal. The following result to this effect is also found in [2] .

Proposition 1. *Let H be a Hilbert space, $\mathbf{\Phi} = H^* = H$, and $\alpha(t) = ct^2$ for $t \in \mathbb{R}$ with a constant $c > 0$. Then (the metric in H is that induced by the scalar product) for every x_0 the function $\alpha \circ d(\cdot, x_0)$ has at an arbitrary $y \in X$ a $\mathbf{\Phi}$-subdifferential ϕ_y at y such that*

$$\alpha(d(z, x_0)) - \alpha(d(y, x_0)) + \phi_y(z) - \phi_y(y) \ge \alpha(d(z, y)) \ , \qquad z \in X. \tag{2}$$

(The statement of the proposition, which in [3] is denoted by $\binom{*}{*}$, is in a sense dual to the condition $(\star)$ found in [2].)

The question arises whether in this case those quadratic functions are the only functions that satisfy inequality (2). To answer the question it is enough to consider (cf. Rolewicz [2]) the case where $X = \mathbb{R}$, $d(x, y) = |x - y|$, and to solve the following. (We now write f instead of α in (2).)

Problem (P) (Rolewicz). *Find all even, nonnegative, and differentiable functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the inequality*

$$(P) \qquad f(t) - f(s) - f'(s)(t - s) \geq f(t - s), \qquad t, s \in \mathbb{R}.$$

REFERENCES

[1] D. Pallaschke and S. Rolewicz, *Foundations of Mathematical Optimization*, Math. Appl. 388, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1997.

[2] S. Rolewicz, *On $\alpha(\cdot)$-monotone multifunctions*, Studia Math., 141 (2000), pp. 263–272.

[3] S. Rolewicz, $\Phi-$*convex functions in metric spaces*, Int. J. Math. Sci. (Kluwer/Plenum), to appear.

Status. We have found a very short proof that every solution is of the form $f(x) = Cx^2$ with a constant $C \geq 0$.