A Short Solution of Rolewicz's Problem

Solution of Problem 01-005 *by* Michael Renardy (Department of Mathematics, Virginia Tech).

By setting $t = s = 0$, we find $f(0) \leq 0$, and since f was assumed nonnegative, $f(0) = 0$. We fix t at a positive, and, respectively, negative value, and let s tend to zero. This yields

$$\lim_{s \to 0} f'(s) = 0,$$

and hence

$$\lim_{s \to 0} \frac{f(s)}{s} = 0.$$

Setting $s = -h$, we find

$$\frac{f(t+h) - f(t)}{h} \leq \frac{-f(h) + f'(h)(t+h)}{h}$$

for $h > 0$ and

$$\frac{f(t+h) - f(t)}{h} \geq \frac{-f(h) + f'(h)(t+h)}{h}$$

for $h < 0$. We let h tend to zero to obtain

$$f'(t) = t \lim_{h \to 0} \frac{f'(h)}{h} =: Ct.$$

Consequently, $f(t) = Ct^2/2$.