Solution of Rolewicz's Problem

Solution of Problem 01-005 *by* Bogdan Choczewski (Faculty of Applied Mathematics, University of Mining and Metallurgy (AGH), Krakow, Poland), Roland Girgensohn (Institute of Biomathematics and Biometry, GSF-Forschungszentrum, Neuherberg, Germany), *and* Zygfryd Kominek (Institute of Mathematics, Silesian University, Katowice, Poland).

1. Problem. We shall solve the following problem.

Problem **(P).** *(Rolewicz). Find all nonnegative and differentiable functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the inequality*

$$f(t) - f(s) - f'(s)(t-s) \geq f(t-s), \qquad t, s \in \mathbb{R} \tag{P}$$

(cf. [2] *and* [4]*, where the problem was originally stated, under the additional assumption that* f *be even).*

It turns out that the assumption is not needed; every solution of Problem (P) is automatically a quadratic function (and therefore even).

We also find all pairs (f, g), $f, g : \mathbb{R} \to \mathbb{R}$, satisfying the functional inequality obtained from (P) by replacing $f'(s)$ by $g(s)$ as well as those which satisfy the related functional equation (without any regularity assumptions on f and g).

2. Solution. We are going to prove the following theorem.

Theorem **1.** *The only solutions* $f : \mathbb{R} \to \mathbb{R}$ *of problem (P) are given by the formula*

$$f(x) = Cx^2, \qquad x \in \mathbb{R}, \tag{S}$$

where C *is a nonnegative constant.*

Proof. If a function $f : \mathbb{R} \to \mathbb{R}$ is a solution to (P), then

$$f(0) = f'(0) = 0$$

(put $t = s = 0$ in (P) to get $f(0) = 0$ and then $s = 0$ in (P) to obtain $f'(0)t \leq 0$, $t \in \mathbb{R}$, yielding $f'(0) = 0$). Thus

$$\lim_{s \to 0} \frac{f(s)}{s} = 0. \tag{1}$$

Denote $h := t - s \in \mathbb{R}$ and rewrite (P) as

$$f'(s) \cdot h \leq f(s+h) - f(s) - f(h), \qquad s, h \in \mathbb{R}. \tag{2}$$

Now assume $s > 0$ to get that

$$\frac{f'(s)}{s}h \leq \frac{f(s+h)-f(h)}{s} - \frac{f(s)}{s}, \qquad h \in \mathbb{R}, s > 0. \tag{3}$$

Thanks to (1), when $s \to 0+$, the RHS tends to $f'(h)$. Thus, the LHS is bounded from above, and

$$2C := \limsup_{s\to 0+} \frac{f'(s)}{s}$$

exists. From (3) we get

$$2Ch \leq f'(h), \qquad h \in \mathbb{R}. \tag{4}$$

Now assume that $s < 0$ in (2), which gives us

$$\frac{f'(s)}{s}h \geq \frac{f(s+h)-f(h)}{s} - \frac{f(s)}{s}, \qquad h \in \mathbb{R}, s < 0.$$

As before, this implies that

$$2D := \liminf_{s\to 0-} \frac{f'(s)}{s}$$

exists and that

$$2Dh \geq f'(h), \qquad h \in \mathbb{R}. \tag{5}$$

Inequalities (4) and (5) together now imply that $Ch \leq Dh$ for all $h \in \mathbb{R}$, and thus $C = D$. Now using (4) and (5) once more, we have $f'(h) = 2Ch$ for all $h \in \mathbb{R}$, and taking $f(0) = 0$ into account we get (S), which was to be proved. □

Remark. The proof of Theorem 1 is due to the second author. Earlier the other authors had proved (S) with the aid of the following proposition.

PROPOSITION **1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be an even, nonnegative, and differentiable function with $f(1) = 1$, satisfying inequality (P). Then we have the following assertions.*

(a) Either f is given by (S) with $C = 1$ or there are an $\varepsilon > 0$ and $a, b \in \mathbb{R}$, $\frac{1}{2} < a < b$, such that $f'(x) > 2x + \varepsilon$, $x \in [a, b]$.

(b) If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to zero, $x_n > 0, n \in \mathbb{N}$, such that $f'(x_n) \geq 2x_n, n \in \mathbb{N}$, then f is given by (S) with $C = 1$.

The first author was able to derive (S) from (P) having additionally assumed that f is even, twice differentiable in a neighborhood of the origin, and it satisfies an initial condition; cf. [1].

3. Pexider-type functional inequality. In connection with (P) let us consider the following inequality:

$$(Q) \qquad f(t) - f(s) - g(s)(t-s) \geq f(t-s), \qquad t, s \in \mathbb{R}.$$

We start with a simple lemma.

LEMMA **1.** *A pair $(f,\ g)$ of functions, each mapping $\mathbb{R}$ into $\mathbb{R}$, where f is differentiable in $\mathbb{R}$, $f(0) = 0$, and g is arbitrary, satisfies inequality (Q) if and only if*

$$(6) \qquad g(t) = f'(t) - f'(0), \qquad t \in \mathbb{R},$$

and f satisfies the inequality

$$(P') \qquad f(t) - f(s) - [f'(s) - f'(0)](t-s) \geq f(t-s), \qquad t, s \in \mathbb{R}.$$

Proof. Let f and g, regular as required, satisfy (Q). For $t > s$ inequality (Q) may be written in the form

$$(7) \qquad \frac{f(t) - f(s)}{t-s} - g(s) \geq \frac{f(t-s)}{t-s}$$

whereas for $t < s$ we have the inequality opposite to (7). Since f is differentiable, we get $f'(s) - g(s) = f'(0)$, which is (6), and f satisfies (P′). The converse implication is obvious. $\square$

Theorem 1 and Lemma 1 together yield the following result.

THEOREM **2.** *If $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative and differentiable function with $f'(0) = 0$, $g : \mathbb{R} \to \mathbb{R}$ is arbitrary, and they both satisfy inequality (Q), then there is a $C \geq 0$ such that*

$$f(t) = C\ t^2\ , \qquad g(t) = 2C\ t, \qquad t \in \mathbb{R}.$$

In the case where f in (Q) is an odd function we have the following theorem.

THEOREM **3.** *A pair $(f,\ g)$ of functions, each mapping $\mathbb{R}$ into $\mathbb{R}$, where f is differentiable in $\mathbb{R}$ and odd, and g is arbitrary, satisfies inequality (Q) if and only if there is a $C \in \mathbb{R}$ such that*

$$(8) \qquad f(t) = C\ t, \qquad g(t) = 0, \qquad t \in \mathbb{R}.$$

Proof. We have $f(0) = 0$ as f is odd. Thus the lemma works. Since now f' in (P$'$) is even, on putting $-s$ in place of s in (P$'$) we get

$$f(t) + f(s) - [f'(s) - f'(0)](t + s) \geq f(t + s), \qquad s, t \in \mathbb{R}.$$

With $t = 0$ here we arrive at $[f'(s) - f'(0)] \cdot s \leq 0, s \in \mathbb{R}$.

On the other hand, with $-t$ in place of t in (P$'$) we obtain

$$[f'(s) - f'(0)](t + s) \geq f(t) + f(s) - f(t + s), \qquad t, s \in \mathbb{R}.$$

Letting $t = 0$ here yields $[f'(s) - f'(0)] \cdot s \geq 0, s \in \mathbb{R}$.

Consequently, $f'(s) = f'(0)$, in turn $f(s) = f'(0)s + B$. But $B = 0$ as f is odd. Finally, by (6), $g(s) = 0, s \in \mathbb{R}$. Thus (8) holds with $C = f'(0)$. The converse implication is obvious. □

4. Pexider-type functional equation. For the functional equation (cf. inequality (Q))

$$(E) \qquad f(t) - f(s) - g(s)(t - s) = f(t - s), \qquad t, s \in \mathbb{R},$$

we have the following result.

THEOREM **4.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions fulfilling equation (E). Then there exist a real constant C and an additive function $a : \mathbb{R} \to \mathbb{R}$ such that*

$$(9) \qquad f(x) = a(x) + Cx^2, \qquad g(x) = 2Cx, \qquad x \in \mathbb{R}.$$

Conversely, the system of functions defined by (9)*, where a is an additive function and $C \in \mathbb{R}$, is a solution of (E).*

Proof. Setting $s = 0$ in (E) we get

$$f(0) = g(0) = 0.$$

Put $t + s$ instead of t in (E) . We have

$$(10) \qquad f(t + s) - f(t) - f(s) = g(s)t, \qquad t, s \in \mathbb{R}.$$

Since the LHS of this equality is symmetric with respect to t and s, so is its RHS. Thus

$$g(s)t = g(t)s, \qquad t, s \in \mathbb{R}.$$

Therefore there exists a constant $C \in \mathbb{R}$ such that $g(x) = 2Cx,\ x \in \mathbb{R}$. Moreover, now (10) has the form

$$(11) \qquad f(t + s) - f(t) - f(s) = 2Cts, \qquad t, s \in \mathbb{R}.$$

We define the function $a : \mathbb{R} \to \mathbb{R}$ by the formula

$$a(x) := f(x) - Cx^2, \qquad x \in \mathbb{R}.$$

According to (11) we obtain $a(t+s) - a(t) - a(s) = 2Cts - C(t+s)^2 + Cs^2 + Ct^2 = 0$ for all $t, s \in \mathbb{R}$, which means that a is an additive function. The other part of the proof is evident. □

Since every Lebesgue measurable additive function $a : \mathbb{R} \to \mathbb{R}$ is linear (cf. [3], for example), Theorem 4 has the following corollary.

COROLLARY **1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function, and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then the pair of function (f, g) is a solution of functional equation (E) if and only if there exist a real constants C and b such that*

$$f(x) = Cx^2 + bx, \qquad g(x) = 2Cx, \qquad x \in \mathbb{R}.$$

REFERENCES

[1] B. CHOCZEWSKI, *Note on a functional-differential inequality*, in Functional Equations - Results and Advances, Z. Daróczy and Zs. Páles, eds., dedicated to the Millennium of The Hungarian State, Kluwer Academic Publishers, Boston/Dordrecht/London, 2001, pp. 21–24.

[2] B. CHOCZEWSKI, R. GIRGENSOHN, Z. KOMINEK, *Rolewicz's Problem*, in Problems & Solutions, SIAM, Philadelphia, 2001; available online from `http://www.siam.org/journals/problems/01-005.htm`.

[3] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Polish Scientific Publishers and Uniwersytet Śląski, Warszawa/Kraków/Katowice, 1985.

[4] S. ROLEWICZ, *On $\alpha(\cdot)$-monotone multifunctions*, Studia Math., 141 (2000), pp. 263–272.