Generalized Amicable Orthogonal Designs: Do They Exist?

Problem 01-006, *by* GIRISH GANESAN AND PETRE STOICA (Uppsala University, Sweden).

A family of $m \times n$ matrices $\{\boldsymbol{B}_i\}_{i=1}^s$ with entries in the set $\{1, 0, -1\}$ and satisfying the conditions

$$\text{(1)} \qquad \boldsymbol{B}_i \boldsymbol{B}_i^T = \boldsymbol{I} \qquad \forall\, i,$$

$$\text{(2)} \qquad \boldsymbol{B}_i \boldsymbol{B}_j^T = -\boldsymbol{B}_j \boldsymbol{B}_i^T, \qquad i \neq j,$$

is a *generalized orthogonal design of order* (m, n) *and size* s.

A family $(\{\boldsymbol{X}_i\}_{i=1}^s, \{\boldsymbol{Y}_i\}_{i=1}^t)$ where the $\boldsymbol{X}_i$'s and $\boldsymbol{Y}_i$'s are $m \times m$ matrices with elements in $\{1, 0, -1\}$ and satisfy the conditions

$$\text{(3)} \qquad \boldsymbol{X}_i \boldsymbol{X}_i^T = \boldsymbol{I},\ \boldsymbol{Y}_i \boldsymbol{Y}_i^T = \boldsymbol{I} \qquad \forall\, i,$$

$$\text{(4)} \qquad \boldsymbol{X}_i \boldsymbol{X}_j^T = -\boldsymbol{X}_j \boldsymbol{X}_i^T, \qquad i \neq j,$$

$$\text{(5)} \qquad \boldsymbol{Y}_i \boldsymbol{Y}_j^T = -\boldsymbol{Y}_j \boldsymbol{Y}_i^T, \qquad i \neq j,$$

$$\text{(6)} \qquad \boldsymbol{X}_i \boldsymbol{Y}_j^T = \boldsymbol{Y}_j \boldsymbol{X}_i^T \qquad \forall\, i, j$$

is an *amicable orthogonal design of order* m *and size* (s, t). In [1] the following result is proved.

THEOREM **1.** *Let* $m = 2^a b$*, where* b *is odd and* $a, b > 0$*. There exists an amicable orthogonal design of order* m *and size* (s, t) *with* $s + t \leq 2a + 2$*, and the bound can be achieved.*

Generalized amicable orthogonal designs, if they exist, are families $\{\boldsymbol{X}_i\}_{i=1}^s$ and $\{\boldsymbol{Y}_i\}_{i=1}^t$ consisting of $m \times n$ matrices that satisfy (3)–(6). The entries are allowed to be arbitrary complex numbers.

(a) Prove that for every m there is an $n \geq m$ for which a generalized orthogonal design exists of order (m, n) and size n.

(b) Is it true that for every m there is an $n \geq m$ for which a generalized amicable orthogonal design $(\{\boldsymbol{X}_i\}_{i=1}^s, \{\boldsymbol{Y}_i\}_{i=1}^t)$ exists with $s + t = 2n$?

REFERENCE

[1] A. V. GERAMITA AND J. SEBERRY, *Orthogonal designs. Quadratic forms and Hadamard matrices*, Lecture Notes in Pure and Appl. Math. 45, Marcel Dekker, New York, 1979.

Status. The proposers have a solution for part (a). Part (b) is open. In other words, the open question is whether or not the recently obtained theory of generalized orthogonal designs extends along the same lines to one of generalized amicable orthogonal designs.