Partial Answer to a Question Concerning Generalized Amicable Orthogonal Designs

Partial Solution to Problem 01-006 *by* Chau Yuen and Yong Liang Guan (Nanyang Technological University, Singapore) and Tjeng Thiang Tjhung (Institute for Infocomm Research, Agency for Science, Technology and Research, Singapore).

Problem summary. Generalized amicable orthogonal designs, if they exist, are families $\{\boldsymbol{X}_j\}_{j=1}^s$ and $\{\boldsymbol{Y}_j\}_{j=1}^t$ consisting of $m \times n$ matrices that satisfy (1) below. The entries are allowed to be arbitrary complex numbers, and H denotes the conjugate transpose.

$$
(1) \qquad
\begin{array}{llr}
\text{(i)} & \boldsymbol{X}_j\boldsymbol{X}_j^H = \boldsymbol{I},\ \boldsymbol{Y}_j\boldsymbol{Y}_j^H = \boldsymbol{I} & \forall\, j, \\
\text{(ii)} & \boldsymbol{X}_j\boldsymbol{X}_k^H = -\boldsymbol{X}_k\boldsymbol{X}_j^H, & j \neq k, \\
\text{(iii)} & \boldsymbol{Y}_j\boldsymbol{Y}_k^H = -\boldsymbol{Y}_k\boldsymbol{Y}_j^H, & j \neq k, \\
\text{(iv)} & \boldsymbol{X}_j\boldsymbol{Y}_k^H = \boldsymbol{Y}_k\boldsymbol{X}_j^H & \forall\, j, k.
\end{array}
$$

(a) Prove that for every m there is an $n \geq m$ for which a generalized orthogonal design exists of order (m, n) and size n.

(b) Is it true that for every m there is an $n \geq m$ for which a generalized amicable orthogonal design $(\{\boldsymbol{X}_j\}_{j=1}^s, \{\boldsymbol{Y}_j\}_{j=1}^t)$ exists with $s + t = 2n$?

In this document, the special case $n = m$ of problem (b) is solved. The case of $n > m$ remains an open issue.

The proposed solution of problem (b) can be obtained through a new orthogonal design called an *amicable complex orthogonal design* (ACOD). Before discussing ACODs, a related orthogonal design called a *complex orthogonal design* (COD) is first reviewed.

Definition 1 **(see [1]).** *A COD of order* n *and type* $(h_1, \dots, h_r)$ *(*h_j *positive integers) on the real commuting variables* $c_1, \dots, c_r$ *is an* $n \times n$ *matrix* $\boldsymbol{C}$*, with entries from* $\{0, \epsilon_1 c_1, \dots, \epsilon_r c_r | \epsilon_j = \pm 1, \pm i\}$ $(i = \sqrt{-1})$ *satisfying*

$$
(2) \qquad \boldsymbol{C}\boldsymbol{C}^H = \left(\sum_{j=1}^{r} h_j\, c_j^2\right) \boldsymbol{I}_n.
$$

Also, $\boldsymbol{C}$ *can be expressed as*

$$
(3) \qquad \boldsymbol{C} = \boldsymbol{Z}_1 c_1 + \cdots + \boldsymbol{Z}_r c_r,
$$

where the $\boldsymbol{Z}_j$ *are* $n \times n$ *matrices with elements* $\{0, \pm 1, \pm i\}$ *satisfying*

$$\text{(4)} \qquad \begin{aligned} &\text{(i)} \quad \boldsymbol{Z}_j \boldsymbol{Z}_j^H = h_j \boldsymbol{I}_n, && 1 \le j \le r, \\ &\text{(ii)} \quad \boldsymbol{Z}_j \boldsymbol{Z}_k^H + \boldsymbol{Z}_k \boldsymbol{Z}_j^H = 0, && 1 \le j \ne k \le r. \end{aligned}$$

THEOREM **1 (see [1, Theorem 4]).** *Let* $\tau(n)$ *denote the maximum number of variables* r *in a COD of order* n*. Then* $\tau(n) \le H(n)$*, where* $H(n) = 2a + 2$ *if* $n = 2^a b$ *with* b *odd.*

Next we define a ACOD by following the approach adopted in [2] to define an *amicable orthogonal design* (AOD) from an *orthogonal design* (OD).

DEFINITION **2.** *Let the matrices* $\boldsymbol{A} = \boldsymbol{X}_1 a_1 + \cdots + \boldsymbol{X}_s a_s$ *and* $\boldsymbol{B} = \boldsymbol{Y}_1 b_1 + \cdots + \boldsymbol{Y}_t b_t$ *be CODs of the same order* n*, where COD* $\boldsymbol{A}$ *is of type* $(f_1, \dots, f_s)$ *on the variables* $\{a_1, \dots, a_s\}$ *and COD* $\boldsymbol{B}$ *is of type* $(g_1, \dots, g_t)$ *on the variables* $\{b_1, \dots, b_t\}$*. Then* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are said to be an ACOD if*

$$\text{(5)} \qquad \boldsymbol{A}\boldsymbol{B}^H = \boldsymbol{B}\boldsymbol{A}^H.$$

A necessary and sufficient condition for an ACOD to exist is that there exists a family of matrices $\{\boldsymbol{X}_1, \dots, \boldsymbol{X}_s; \boldsymbol{Y}_1, \dots, \boldsymbol{Y}_t\}$ satisfying

$$\text{(6)} \qquad \begin{aligned} &\text{(i)} \quad \boldsymbol{X}_j \boldsymbol{X}_j^H = f_j \boldsymbol{I}_n, && 1 \le j \le s, \\ &\phantom{\text{(i)}} \quad \boldsymbol{Y}_k \boldsymbol{Y}_k^H = g_k \boldsymbol{I}_n, && 1 \le k \le t, \\ &\text{(ii)} \quad \boldsymbol{X}_j \boldsymbol{X}_k^H + \boldsymbol{X}_k \boldsymbol{X}_j^H = 0, && 1 \le j \ne k \le s, \\ &\text{(iii)} \quad \boldsymbol{Y}_j \boldsymbol{Y}_k^H + \boldsymbol{Y}_k \boldsymbol{Y}_j^H = 0, && 1 \le j \ne k \le t, \\ &\text{(iv)} \quad \boldsymbol{X}_j \boldsymbol{Y}_k^H = \boldsymbol{Y}_k \boldsymbol{X}_j^H, && 1 \le j \le s,\ 1 \le k \le t, \end{aligned}$$

where $\boldsymbol{X}_j$ and $\boldsymbol{Y}_k$ are all $\{0, \pm 1, \pm i\}$ matrices of order n.

It should be noted that (6) corresponds to the square case ($n = m$) of (1). Hence the existence and an upper bound on the maximum number of variables $s+t$ of an ACOD would help to solve problem (b) stated above.

PROPOSITION **1.** *Assume that the CODs* $\boldsymbol{A} = \boldsymbol{X}_1 a_1 + \cdots + \boldsymbol{X}_s a_s$ *and* $\boldsymbol{B} = \boldsymbol{Y}_1 b_1 + \cdots + \boldsymbol{Y}_t b_t$ *as defined in Definition* 2 *exist. By letting*

$$\text{(7)} \qquad \begin{aligned} \boldsymbol{Z}_j &= \boldsymbol{X}_j, \quad 1 \le j \le s, \\ \boldsymbol{Z}_{s+k} &= i\boldsymbol{Y}_k, \quad 1 \le k \le t, \quad i = \sqrt{-1}, \end{aligned}$$

a COD $\boldsymbol{C} = \boldsymbol{Z}_1 c_1 + \cdots + \boldsymbol{Z}_r c_r$ *with* $r = s + t$ *variables of type* $(f_1, \dots, f_s, g_1, \dots, g_t)$ *will be formed by* $\boldsymbol{Z}_j$ $(1 \le j \le s + t)$*.*

Proof. It can be shown that (7) satisfies all the constraints of (4). For example, 4 (ii) can be verified as follows:

$$\begin{aligned}\boldsymbol{Z}_j\boldsymbol{Z}_{s+k}^H + \boldsymbol{Z}_{s+k}\boldsymbol{Z}_j^H &= \boldsymbol{X}_j(i\boldsymbol{Y}_k)^H + (i\boldsymbol{Y}_k)\boldsymbol{X}_j^H \\ &= -i\boldsymbol{X}_j\boldsymbol{Y}_k^H + i\boldsymbol{Y}_k\boldsymbol{X}_j^H \\ &= 0, \qquad 1 \le j \le s,\ 1 \le k \le t.\end{aligned}$$

□

PROPOSITION **2.** *The total number of variables $s+t$ of an ACOD is bounded above by $H(n)$.*

Proof. From Proposition 1 it is clear that whenever an ACOD with $s+t$ variables exists, a COD with $r = s+t$ variables will exist. Furthermore, it was been stated in Theorem 1 that the maximum number of variables of a COD is bounded by $H(n)$, that is, $\max(r) \le H(n)$. Hence the maximum total number of variables of an ACOD is also bounded by $H(n)$, that is, $\max(s+t) \le H(n) = 2a+2$ if $n = 2^a b$ with b odd. As a result, Proposition 2 is proved. □

Proposition 2 proves that when $n = m$ we have $\max(s+t) = 2a+2$, where $n = 2^a b$ and b is odd. Since $n > a+1$, we have $\max(s+t) < 2n$, so for the case $n = m$, a generalized amicable orthogonal design $(\{\boldsymbol{X}_j\}_{j=1}^s, \{\boldsymbol{Y}_j\}_{j=1}^s)$ does not exist with $s+t = 2n$. This answers the question in part (b) of the problem.

REFERENCES

[1] A. V. GERAMITA AND J. M. GERAMITA, *Complex orthogonal designs*, J. Combin. Theory Ser. A, 25 (1978), pp. 211–225.

[2] W. WOLFE, *Amicable orthogonal designs—existence*, Canad. J. Math., 28 (1976), pp. 1006–1020.