All in the Family

Solution of Problem 01-001 *by* DAVID BORWEIN (University of Western Ontario, London, ON, Canada).

For an integer $N > 0$, let

$$Q_N := \sum_{n=-N}^{\infty} \frac{n^n e^{-n}}{(N+n)!}.$$

On expanding e^{-n} in powers of n and then changing the order of summation, we obtain

$$\begin{aligned} Q_N &= \sum_{n=-N}^{\infty} \frac{n^n}{(N+n)!} \sum_{k=n}^{\infty} \frac{(-n)^{k-n}}{(k-n)!} = \sum_{k=-N}^{\infty} (-1)^k \sum_{n=-N}^{k} \frac{(-1)^n n^k}{(N+n)!(k-n)!} \\ &= \sum_{k=-N}^{-1} (-1)^k \sum_{n=-N}^{k} \frac{(-1)^n n^k}{(N+n)!(k-n)!}, \end{aligned}$$

since, for integral $k \geq 0$,

$$\begin{aligned} \sum_{n=-N}^{k} \frac{(-1)^n n^k}{(N+n)!(k-n)!} &= \frac{1}{(N+k)!} \sum_{n=-N}^{k} \binom{N+k}{N+n} (-1)^n n^k \\ &= \frac{(-1)^N}{(N+k)!} \sum_{n=0}^{N+k} \binom{N+k}{n} (-1)^n (n-N)^k = 0. \end{aligned}$$

The final identity is a consequence of the familiar result

$$\sum_{n=0}^{m} (-1)^n \binom{m}{n} n^r = 0 \text{ for } r = 0, 1 \ldots, m-1,$$

which can easily be verified by observing that

$$\left(\frac{d}{dx}\right)^r (1-e^x)^m = \sum_{n=0}^{m} (-1)^n \binom{m}{n} n^r e^{nx} = 0 \text{ when } x = 0 \text{ and } 0 \leq r < m.$$

It follows that

$$\begin{aligned} Q_N &= \sum_{k=1}^{N} \sum_{n=k}^{N} \frac{(-1)^n n^{-k}}{(N-n)!(n-k)!} = \sum_{n=1}^{N} \frac{(-1)^n}{(N-n)!} \sum_{k=1}^{n} \frac{n^{-k}}{(n-k)!} \\ &= \sum_{n=1}^{N} \frac{(-1)^n}{(N-n)! n^n} \sum_{k=0}^{n-1} \frac{n^k}{k!}, \end{aligned}$$

a rational number, and hence that

$$\begin{aligned} S_N &= Q_N - \sum_{n=-N}^{-1} \frac{n^n e^{-n}}{(N+n)!} = Q_N - \sum_{n=1}^{N} \frac{(-1)^n e^n}{(N-n)! n^n} \\ &=: P_N(e), \end{aligned}$$

as desired.