An Eulerian Approach

Solution of Problem 01-001 *by* VINICIUS-NICOLAE-PETRE ANGHEL (AECL Chalk River Laboratory, Chalk River, ON, Canada).

The solution to this problem is found by analyzing for $-1 \le x \le 1$ the generalized series

$$S_N(x) = \sum_{k=0}^{\infty} \frac{k^k x^{k+N}}{(k+N)!\, e^k}. \tag{1}$$

By Stirling's formula, $\sum_{k=0}^{\infty} k^k/((k+N)!e^k)$ converges for $N \ge 1$, and the sum is given by $\lim_{x \to 1-} S_N(x)$ in view of Abel's theorem on power series. The sequence $(S_N(x))$ satisfies

$$S_{N+1}(x) = \int_0^x S_N(y)\, dy. \tag{2}$$

As the starting point for applying (2), we take

$$S_0(x) = \sum_{k=0}^{\infty} \frac{k^k x^k}{k!\, e^k}, \qquad -1 < x < 1. \tag{3}$$

It is well known that for $|x| < e^{-1}$, the solution of $x = w/\exp(w)$ is given by

$$w(x) = \sum_{k=1}^{\infty} \frac{k^{k-1} x^k}{k!}. \tag{4}$$

Note. This is a textbook example of the Lagrange inversion formula. See [1, p. 146 and p. 348], where (4) is attributed to Euler. Thus if $u = u(x)$ satisfies $1 - u = xe^{-u}$, then

$$1 - u(x) = \sum_{k=1}^{\infty} \frac{k^{k-1} x^k}{k!\, e^k},$$

so

$$S_0(x) = 1 - x\frac{du}{dx} = 1 + \frac{x}{ue^u} = 1 + \frac{1-u}{u} = \frac{1}{u}.$$

For $N \ge 0$, define G_N by $G_N(u(x)) = S_N(x)$. Then from (2) we find that $(G_N(u))$ satisfies

$$G_{N+1}(u) = \int_u^1 te^t\, G_N(t)\, dt. \tag{5}$$

The desired series sum is $S_N(1) = G_N(0)$ with $N \ge 1$. From (5) and $G_0(u) = 1/u$, we obtain

$$G_1(u) = \int_u^1 te^t \cdot \frac{1}{t}\, dt = e - e^u$$

and

$$G_2(u) = \int_u^1 te^t(e - e^t)\,dt = -\tfrac{1}{4}e^2 + e^u \cdot e - \tfrac{1}{4}e^{2u} + u(-e^u \cdot e + \tfrac{1}{2}e^{2u}).$$

Thus

$$S_1 = e - 1 \qquad \text{and} \qquad S_2 = -\tfrac{1}{4}e^2 + e - \tfrac{1}{4}.$$

Small case results for $G_N(u)$ lead one to conjecture the following result.

THEOREM **1.** *For $N \geq 1$, the functions $G_N(u)$ are given by*

$$G_N(u) = \sum_{k=0}^{N-1} u^k \sum_{l=0}^{N} c_{kl}\, e^{lu} e^{N-l}, \tag{6}$$

where the coefficients c_{kl} are all rational.

Proof. This theorem is proven by induction. The cases $N = 1$ and $N = 2$ are already established. We assume the theorem true for N and prove it for $N + 1$.

Thus (5) gives

$$G_{N+1}(u) = \int_u^1 te^t \left(\sum_{k=0}^{N-1} t^k \sum_{l=0}^{N} c_{kl}\, e^{lt} e^{N-l} \right) dt.$$

To complete the proof, it is enough to consider the contributions to $G_{N+1}(u)$ from the term $c_{kl}\, u^k e^{lu} e^{N-l}$ in the expansion of $G_N(u)$ and to verify that each one is a rational multiple of $u^j e^{mu} e^{N+1-m}$ for some $j \leq N$ and $m \leq N + 1$. By elementary calculus,

$$\int_u^1 t^n e^{at}\,dt = \frac{P_n(a)e^a - P_n(au)e^{au}}{a^{n+1}}, \tag{7}$$

where

$$P_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} j!\, x^{n-j}.$$

The contribution to $G_{N+1}(u)$ from $c_{kl}\, u^k e^{lu} e^{N-l}$ is

$$c_{kl}\, e^{N-l} \int_u^1 t^{k+1} e^{(l+1)t}\,dt.$$

Note that from (7) and the fact that P_n has integer coefficients, each resulting term is either a rational multiple of $u^j e^{mu} e^{(N+1)-m}$ for some $j \leq N$ and $1 \leq m = l + 1 \leq N + 1$ or else a rational multiple of $e^{l+1} e^{N-l} = e^{N+1}$. The latter contributes to the constant term ($k = l = 0$) of the expansion for $G_{N+1}(u)$. The inductive proof is thus complete. $\square$

Finally,

$$S_N = G_N(0) = \sum_{l=0}^{N} c_{0l}\, e^{N-l} = P_N(e),$$

a rational polynomial in e of degree N.

Editorial note. The title is chosen by the editor. The use of "Eulerian" is not in any technical sense. Rather, it is to acknowledge that the starting point is a result that was known to Euler and that the spirit of the approach is reminiscent of the great mathematician.

REFERENCE

[1] G. Pólya and G. Szegö, *Problems and Theorems in Analysis* I, Springer-Verlag, Berlin, Heidelberg, 1998.