How the Discovery was Made

Solution of Problem 01-001 *by* Jonathan Borwein (CECM, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada (jborwein@cecm.sfu.ca).

It is an easy matter, for any given small N, to pick off the closed form of the nonconstant coefficients in P_N. Indeed, one can be led to discover from $N < 9$, say, that

$$(1) \qquad Q_N \; := \; \sum_{n=0}^{\infty} \frac{n^n e^{-n}}{(n+N)!} - \sum_{k=1}^{N} \frac{(-1)^{k-1} e^k}{(N-k)!\, k^k}$$

is a rational number for each integer $N > 0$. The first four constant coefficients are $-1, -1/4, -7/108$, $-97/6912$. More delicately, one can ultimately discover that

$$(2) \qquad Q_N \; = \; \sum_{k=1}^{N} \frac{(-1)^k}{(N-k)! k^k} \sum_{n=0}^{k-1} \frac{k^n}{n!}.$$

This relies on replacing $\exp(-n)$ by its series, exchanging the order of summation, and then discovering and deriving the identity

$$(3) \qquad \sum_{k=0}^{n} (-1)^k \, k^n \binom{N+n}{N+k} \; = \; (-1)^n \sum_{k=1}^{N} k^n (-1)^{k-1} \binom{N+n}{N-k},$$

or, equivalently,

$$\sum_{k=0}^{M+N} (-1)^k \, (M-k)^M \binom{M+N}{k} = 0$$

for all $M, N > 0$. This in turn follows, on using the binomial theorem, from

$$(4) \qquad \sum_{k=0}^{P} (-1)^k \, k^n \binom{P}{k} = 0$$

for all $0 \leq n < P$. The final identity (4) is established by setting $Df(x) := xf'(x)$ and observing that $D^k (1-x)^P$ has a zero at 1 for $k < P$.

Actually, in the derivation of (3) I was assisted by Salvy and Zimmermann's "gfun" lovely Maple package which hunts for generating functions and recursions for sequences of integers. Using their program, I found the closed form for $N < 4$ in a somewhat obscure form. I then used linear algebra to interpolate the pattern for a few more cases and discovered (3).

Editorial note. There are several quick proofs of (4). For a combinatorial proof, use the method of inclusion-exclusion to count surjections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, P\}$.