

## A Problem from Blacksburg Middle School

*Problem 02-005, by MICHAEL RENARDY (Virginia Tech).*

My daughter's 7th grade class was assigned the following problem: Find as many different ways as possible to make change for 50 cents. Apparently neither the students nor the teacher were expected to come up with a procedure to determine all possible ways. The problem stated that there were more than 75 such ways, and the following would never have been written if this claim were correct.

Actually, there are precisely 50 ways to make change for 50 cents. I join my daughter in finding this coincidence amazing, which motivates the following definition.

**DEFINITION.** For any natural number  $i$ , let  $C(i)$  denote the number of different ways to make change for  $i$  cents (using United States coins of 1, 5, 10, 25, 50, and 100 cents). We call  $i$  amazing if  $C(i) = i$ .

- (a) Program an algorithm to compute  $C(i)$ .
- (b) What is the asymptotic behavior of  $C(i)$  as  $i \rightarrow \infty$ ?
- (c) Show that the only amazing numbers are 1 and 50.

Of course, the definition of an amazing number depends on the denominations of available coins. The only Euro-amazing numbers are 1 and 2. If you make coins of 1, 2, and 3 cents, but not 4 and 5 cents, then 1, 2, 3, 4, and 5 are all amazing.

- (d) Show that for any given number, you can find a set of coin denominations which makes it amazing.

Observations like these motivate the following extension to the problem:

After the state cuts your institution's budget for the fifth time in a year, you decide to take the latest buyout offer and retire. To do something meaningful with the rest of your life, you acquire a remote uninhabited island and declare it an independent country. After your parliament unanimously approves a constitution which outlaws budget cuts to universities, you go about the essentials of government such as minting coins. You want to create a truly amazing currency based on sound mathematical principle, which means, of course, that you want to have as many amazing numbers of possible. Your enthusiasm is only slightly dampened by the observation that the number of amazing numbers is necessarily finite.

- (e) Is there an upper bound? If yes, how is it achieved?

*Status.* Part (e) is open.