Solving a Problem from Blacksburg Middle School

In Problem 02-005, Michael Renardy (Virginia Tech) raises some questions about counting the ways of giving change. In particular, when is the number of ways equal to the monetary amount relative to the given coinage system? Numbers satisfying this condition are then said to be amazing. For example, if the set of coin values is $\{1, 5, 10, 25, 50, 100\}$, there are 50 ways to give change for 50. So for the USA coinage system, 50 is amazing. The full text of the problem can be found at

`http://www.siam.org/journals/problems/downloadfiles/02-005.pdf.`

Solution by Jonas Degrave[1] (Wijtschate, Belgium). Let us start by giving a good definition for the function $f_S(k)$, which gives us the total number of possibilities for paying a value k in a currency with denomination set S. For example, the denomination set for the Euro would be $\{1, 2, 5, 10, 20, 50, 100\}$.

Definition **1** (Number of ways to make change)**.** *$f_S(k)$ for a set S of size d with $S_i \in \mathbb{N}$ is equal to the number of lists $(a_1, a_2, a_3, \ldots, a_d)$ $(a_n \in \mathbb{N})$ for which $\sum_{i=1}^{d} a_i.S_i = k$. Define*

$$f_\emptyset(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We can derive some trivial equations from this definition.

$$f_S(0) \equiv 1, \tag{1}$$

$$f_S(k) = \begin{cases} f_S(k) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases} \tag{2}$$

$$f_{\{1\}}(k) = \begin{cases} 0 & \text{if } k < 0, \\ 1 & \text{if } k \geq 0, \end{cases} \tag{3}$$

$$f_{\{c\}}(k) = \begin{cases} 1 & \text{if } k|c \text{ and } k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

From Definition 1 we can extract a trivial yet very effective rule to calculate the number of possibilities to make change.

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Theorem 1 (Summation rule)**.** $f_S(k) = \sum_{n_1=0}^{+\infty} f_{S\setminus\{S_1\}}(k - S_1.n_1)$.

This is easy to see, as the sum adds all possibilities if you hold a_1. You could, however, state an upper bound, and the smallest upper bound is $\lfloor \frac{k}{S_1} \rfloor$, because if n_1 gets bigger than $\lfloor \frac{k}{S_1} \rfloor$, $k - S_1.n_1$ becomes smaller than 0. By (4) we have already seen that this means $f_{S\setminus\{S_1\}}(k - S_1.n_1) = 0$.

Theorem 2 (Bounded Summation rule)**.** $f_S(k) = \sum_{n_1=0}^{\lfloor \frac{k}{S_1} \rfloor} f_{S\setminus\{S_1\}}(k - S_1.n_1)$.

This simple theorem can be expanded when applied several times. It is also possible for the upper bound of the summation to be higher than the ones used here. Note that what I call S_1 can be any element of the set, not equal to S_2 which is another element of S. Call $d = |S|$.

$$f_S(k) = \sum_{n_1=0}^{\left\lfloor \frac{k}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.S_1}{S_2} \right\rfloor} f_{S\setminus\{S_1,S_2\}}(k - (n_1.S_1) - (n_2.S_2)), \tag{5}$$

$$f_S(k) = \sum_{n_1=0}^{\left\lfloor \frac{k}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.S_1}{S_2} \right\rfloor} \cdots \sum_{n_{d-1}=0}^{\left\lfloor \frac{k-\sum_{i=0}^{d-2} n_i.S_i}{S_{d-1}} \right\rfloor} f_{\{S_d\}}\left(k - \sum_{i=0}^{d-1} n_i.S_i\right), \tag{6}$$

$$f_S(k) = \sum_{n_1=0}^{\left\lfloor \frac{k}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.S_1}{S_2} \right\rfloor} \cdots \sum_{n_d=0}^{\left\lfloor \frac{k-\sum_{i=0}^{d} n_i.S_i}{S_{d-1}} \right\rfloor} f_{\emptyset}\left(k - \sum_{i=0}^{d} n_i.S_i\right). \tag{7}$$

From this last one we can extract the following theorem.

Theorem 3. *For a set $S, 0 < S_i \in \mathbb{N}$ $(\forall i \in \{1, 2, \ldots, d\}), \forall k \in \mathbb{Z}, S_j \in S :$ $f_S(k + S_j) = f_{S\setminus\{S_j\}}(k + S_j) + f_S(k)$ and therefore $f_S(k + S_j) \geqslant f_S(k)$.*

Proof.

$$f_S(k+S_d) = \sum_{n_1=0}^{\left\lfloor \frac{k+S_d}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k+S_d-n_1.S_1}{S_2} \right\rfloor} \cdots$$

$$\cdots \sum_{n_d=0}^{\left\lfloor \frac{k+S_d-\sum_{i=0}^{d} n_i.S_i}{S_d} \right\rfloor} f_\emptyset\left(k+S_d-\sum_{i=0}^{d} n_i.S_i\right),$$

$$f_S(k+S_d) = f_{S\setminus\{S_d\}}(k+S_d) + \sum_{n_1=0}^{\left\lfloor \frac{k+S_d}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k+S_d-n_1.S_1}{S_2} \right\rfloor} \cdots$$

$$\cdots \sum_{n_d=1}^{\left\lfloor \frac{k+S_d-\sum_{i=0}^{d-1} n_i.S_i}{S_d} \right\rfloor} f_\emptyset\left(k+S_d-\sum_{i=0}^{d} n_i.S_i\right),$$

$$f_S(k+S_d) = f_{S\setminus\{S_d\}}(k+S_d) + \sum_{n_1=0}^{\left\lfloor \frac{k+S_d}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k+S_d-n_1.S_1}{S_2} \right\rfloor} \cdots$$

$$\cdots \sum_{n_d=1}^{\left\lfloor \frac{k-\sum_{i=1}^{d-1} n_i.S_i}{S_d} \right\rfloor+1} f_\emptyset\left(k+S_d-\sum_{i=0}^{d} n_i.S_i\right),$$

$$f_S(k+S_d) = f_{S\setminus\{S_d\}}(k+S_d) + \sum_{n_1=0}^{\left\lfloor \frac{k+S_d}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k+S_d-n_1.S_1}{S_2} \right\rfloor} \cdots$$

$$\cdots \sum_{n_d=0}^{\left\lfloor \frac{k-\sum_{i=1}^{d-1} n_i.S_i}{S_d} \right\rfloor} f_\emptyset\left(k - \sum_{i=0}^{d} n_i.S_i\right),$$

$$f_S(k+S_d) = f_{S\setminus\{S_d\}}(k+S_d) + f_S(k),$$

$$f_S(k+S_d) \geqslant f_S(k).$$

□

An algorithm to calculate $f_S(k)$. Here is the algorithm I have used to calculate the number of possibilities to make change for k with a denomination set S. It is a direct application of (4) and (6). To use American currency, set $S = \{1, 5, 10, 25, 50, 100\}$.

```
Data: integer k, set S
Result: numfound: the number of possibilities to make change for
begin
    permlist ← {k}
    coins ← sortAscending(S)
    for i from numberOfElementsOf(coins) to 2 by −1 do
        newpermlist ← ∅
        for j from 1 to numberOfElementsOf(permlist) by 1 do
            a ← 0
            while permlist[j] − a ∗ coins[i] ⩾ 0 do
                add {permlist[j] − a ∗ coins[i]} to newpermlist
                add 1 to a
        permlist ← newpermlist
    numfound = 0
    for i from 1 to numberOfElementsOf(permlist) do
        if permlist[i] is dividable by coins[1] then
            add 1 to numfound
end
```

Calculating some $f_S(k)$ explicitly for small S. We have already examined the explicit function for $|S| = 1$ [see (4)]. Now we are going to try to find a simple explicit function for $S = \{1, z\}$ with $1 < z \in \mathbb{N}$.

Explicit function for $f_{\{1,z\}}(k)$. From (6),

$$f_S(k) = \sum_{n_1=0}^{\lfloor \frac{k}{z} \rfloor} f_{\{1\}}(k - z.n_1),$$

$$f_S(k) = \sum_{n_1=0}^{\lfloor \frac{k}{z} \rfloor} 1,$$

$$f_S(k) = \begin{cases} \left\lfloor \frac{k}{z} \right\rfloor + 1, & k \geqslant 0, \\ 0, & k < 0. \end{cases} \tag{8}$$

□

Well, that went all right. So, why not try something harder, like an explicit function for $S = \{1, z, 2z\}$ with $1 < z \in \mathbb{N}$?

Explicit function for $f_{\{1,z,2z\}}(k)$.

$$f_S(k) = \sum_{n_1=0}^{\lfloor \frac{k}{2z} \rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.2z}{z} \right\rfloor} f_{\{1\}}(k - 2.z.n_1 - z.n_2).$$

Hmm, that's quite hard... but remark that from Theorem 3 we can see that if $\lfloor z.j \rfloor \leqslant k < \lfloor z.j \rfloor + z$, $f_S(k) = f_S(\lfloor z.j \rfloor)$, $j \in \mathbb{N}$. So we might make this equation a bit easier by searching for $f_S(z.k)$.

$$f_S(z.k) = \sum_{n_1=0}^{\lfloor \frac{zk}{2z} \rfloor} \sum_{n_2=0}^{\left\lfloor \frac{zk-n_1.2z}{z} \right\rfloor} f_{\{1\}}(zk - 2.z.n_1 - z.n_2),$$

$$f_S(z.k) = \sum_{n_1=0}^{\lfloor \frac{k}{z} \rfloor} k - 2.n_1 + 1.$$

This is a summation over a simple arithmetic progression.

$$f_S(z.k) = \begin{cases} \lfloor \frac{k}{2} \rfloor.k + k - (\lfloor \frac{k}{2} \rfloor)^2 + 1, & k \geqslant 0, \\ 0, & k < 0. \end{cases}$$

To solve this we will need to separate between odd and even z.

$$f_S(z.k) \quad = \quad \begin{cases} \frac{k^2}{4}+k+1 & \text{if } k \text{ even and } k \geqslant 0, \\ \frac{k^2}{4}+k+\frac{3}{4} & \text{if } k \text{ odd and } k \geqslant 0, \\ 0 & \text{if } k < 0 \end{cases}$$

or

$$f_S(m) = \begin{cases} \frac{k^2}{4}+k+1 & \text{if } k \text{ even and } k = \lfloor \frac{m}{z} \rfloor \geqslant 0, \\ \frac{k^2}{4}+k+\frac{3}{4} & \text{if } k \text{ odd and and } k = \lfloor \frac{m}{z} \rfloor \geqslant 0, \\ 0, & k < 0. \end{cases} \tag{9}$$

□

Hmm, hard enough. I don't believe we need to try harder sets.

Determining the asymptotic behavior of $f_S(k)$ at $+\infty$. Now, we have all we need to estimate the asymptotic behavior of any set S at $+\infty$. First of all, we are going to find an upper bound for a set S.

Proof. We'll start from (6).

$$f_S(k) \quad = \sum_{n_1=0}^{\left\lfloor \frac{k}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.S_1}{S_2} \right\rfloor} \cdots \sum_{n_{d-1}=0}^{\left\lfloor \frac{k-\sum_{i=0}^{d-1} n_i.S_i}{S_{d-1}} \right\rfloor} f_{\{S_d\}}\left(k - \sum_{i=0}^{d-1} n_i.S_i\right).$$

We will replace $f_{\{S_d\}}\left(k - \sum_{i=0}^{d-1} n_i.S_i\right)$ by its maximum: 1 [see (4)].

$$f_S(k) \quad \leqslant \sum_{n_1=0}^{\left\lfloor \frac{k}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.S_1}{S_2} \right\rfloor} \cdots \sum_{n_{d-1}=0}^{\left\lfloor \frac{k-\sum_{i=0}^{d-1} n_i.S_i}{S_{d-1}} \right\rfloor} 1.$$

We replace again by the maximum, namely, when $n_i = 0$
for $i \in \{1, 2, \dots, d-2\}$.

$$f_S(k) \;\leqslant\; \sum_{n_1=0}^{\left\lfloor \frac{k}{S_1} \right\rfloor} \sum_{n_2=0}^{\left\lfloor \frac{k-n_1.S_1}{S_2} \right\rfloor} \cdots \sum_{n_{d-2}=0}^{\left\lfloor \frac{k-\sum_{i=0}^{d-2} n_i.S_i}{S_{d-2}} \right\rfloor} \left\lfloor \frac{k+1}{S_{d-1}} \right\rfloor.$$

Keep on replacing by the maximum, until you get this equation.

$$f_S(k) \;\leqslant\; \prod_{i=0}^{d-1} \left\lfloor \frac{k+1}{S_i} \right\rfloor,$$

$$f_S(k) \;\leqslant\; \prod_{i=0}^{d-1} \frac{k+1}{S_i},$$

$$f_S(k) \;\leqslant\; \frac{(k+1)^{d-1}}{\prod_{i=0}^{d-1} S_i}.$$

□

That's an upper bound! The lower bound is harder, because there are sets for which $f_S(k)$ is zero in certain areas. If S contains the number 1, however, it's quite easy.

Proof. We'll start from (6) but with upper bound $+\infty$.

$$f_S\left(k.\sum_{i=0}^{d} S_i\right) \;=\; \sum_{n_1=0}^{+\infty} \sum_{n_2=0}^{+\infty} \cdots \sum_{n_{d-1}=0}^{+\infty} f_{\{1\}}\left(k.\sum_{i=0}^{d} S_i - \sum_{i=0}^{d-1} n_i.S_i\right),$$

$$f_S\left(k.\sum_{i=0}^{d} S_i\right) \;\geqslant\; \sum_{n_1=0}^{k} \sum_{n_2=0}^{k} \cdots \sum_{n_{d-1}=0}^{k} f_{\{1\}}\left(\underbrace{k.\sum_{i=0}^{d} S_i - \sum_{i=0}^{d-1} n_i.S_i}_{>0}\right),$$

$$f_S\left(k.\sum_{i=0}^{d} S_i\right) \;\geqslant\; \sum_{n_1=0}^{k} \sum_{n_2=0}^{k} \cdots \sum_{n_{d-1}=0}^{k} 1,$$

$$f_S\left(k.\sum_{i=0}^{d} S_i\right) \;\geqslant\; k^{d-1}.$$

□

Lower bound! We can, however, try to extend this for any set S.

Proof. We'll start from (6) but with upper bound $+\infty$.

$$f_S\left(k.\sum_{i=0}^{d-1} S_i.S_d\right) \;=\; \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{d-1}=0}^{\infty} f_{S_d}\left(k.\sum_{i=0}^{d} S_i.S_d-\sum_{i=0}^{d-1} n_i.S_i\right).$$

It seems hard to find a lower bound for this, since it can be very irregular. We need to go back to (4).

(i) Is $S_d|k.\sum_{i=0}^{d} S_i.S_d-\sum_{i=0}^{d-1} n_i.S_i$?
This is the case for at least the cases when $n_i = l_i.S_d : l_i = 0\dots k$, since every term has a coefficient S_d.

(ii) Is $k.\sum_{i=0}^{d} S_i.S_d-\sum_{i=0}^{d-1} n_i.S_i \geqslant 0$ ($\forall n_i : l_i.S_d : l_i = 0\dots k$)?

$$k.\sum_{i=0}^{d} S_i.S_d-\sum_{i=0}^{d-1} n_i.S_i \geqslant k.\sum_{i=0}^{d} S_i.S_d-\sum_{i=0}^{d-1} k.S_d.S_i > 0.$$

So we have at least $(k+1)^{d-1}$ cases where $f_{S_d} = 1$.

$$f_S\left(k.\sum_{i=0}^{d-1} S_i.S_d\right) \;\geqslant\; (1+k)^{d-1}.$$

□

We have both a lower bound and an upper bound, so in the big-Oh notation $O(f_S(k)) = k^{d-1}$. However, this doesn't immediately mean that for ANY k, $f_S(k) \sim k^{d-1}$. Fortunately, this is the case in the American currency. A first step in determining the asymptotic behavior will be to calculate an explicit function for $f_S(100.k)$, when S is the American denomination set, $S = \{1, 5, 10, 25, 50, 100\}$.

Proof. We begin with (6). We pick $S_1 = 100$, $S_2 = 50$, $S_3 = 25$, $S_4 = 10$, $S_5 = 5$, $S_6 = 1$.

$$f_S(100.k) = \sum_{n_1=0}^{\left\lfloor\frac{100.k}{100}\right\rfloor}\;\sum_{n_2=0}^{\left\lfloor\frac{100.k-n_1.100}{50}\right\rfloor}\;\sum_{n_3=0}^{\left\lfloor\frac{100.k-n_1.100-n_2.50}{25}\right\rfloor}\;\sum_{n_4=0}^{\left\lfloor\frac{100.k-n_1.100-n_2.50-n_3.25}{10}\right\rfloor}\cdots$$

$$\cdots\sum_{n_5=0}^{\left\lfloor\frac{100.k-n_1.100-n_2.50-n_3.25-n_4.10}{5}\right\rfloor} f_{\{1\}}\left(100.k-\sum_{i=0}^{5} n_i.S_i\right),$$

$$f_S(100.k) = \sum_{n_1=0}^{k} \sum_{n_2=0}^{2.k-2.n_1} \sum_{n_3=0}^{4.k-4.n_1-2.n_2} \sum_{n_4=0}^{10.k-10.n_1-5.n_2-\lfloor \frac{n_3.25}{10} \rfloor} \cdots$$
$$\cdots \sum_{n_5=0}^{20.k-20.n_1-10.n_2-5.n_3-2.n_4} 1,$$

$$f_S(100.k) = \sum_{n_1=0}^{k} \sum_{n_2=0}^{2.k-2.n_1} \sum_{n_3=0}^{4.k-4.n_1-2.n_2} \cdots$$
$$\cdots \sum_{n_4=0}^{10.k-10.n_1-5.n_2-\lfloor \frac{n_3.25}{10} \rfloor} 20.k - 20.n_1 - 10.n_2 - 5.n_3 - 2.n_4 + 1.$$

To solve this equation, even Maple$^{\text{TM}}$ needs help, so we will separate n_3 in even and in odd cases.

$$f_S(100.k) = \sum_{n_1=0}^{k} \sum_{n_2=0}^{2.k-2.n_1}$$
$$\Big(\Big(\sum_{n_3=0}^{2.k-2.n_1-n_2+1} \sum_{n_4=0}^{10.k-10.n_1-5.n_2-5.n_3} 20.k - 20.n_1 - 10.n_2 - 10.n_3 - 2.n_4 + 1\Big)$$
$$+\Big(\sum_{n_3=1}^{2.k-2.n_1-n_2+1} \sum_{n_4=0}^{10.k-10.n_1-5.n_2-5.n_3+3} 20.k - 20.n_1 - 10.n_2 - 10.n_3 - 2.n_4 + 6\Big)\Big),$$

which, according to Maple$^{\text{TM}}$, is equal to

$$f_S(100.k) = 1 + \frac{127}{6}.k + \frac{161}{2}.k^2 + 112.k^3 + 65.k^4 + \frac{40}{3}.k^5. \tag{10}$$

□

Now, we have one final step to prove the behavior of $f_S(k)$ at $+\infty$ for the American denomination set.

Behavior for $f_S(k)$ of the American denomination set at $+\infty$. We know from Theorem 3 that

$f_S(k+1) \geqslant f_S(k)$. Take $i \in \{0 \dots 99\}$.

$$\begin{aligned}
\frac{f_S(100.k)}{\frac{40}{3}.k^5} &\leqslant \frac{f_S(100.k+i)}{\frac{40}{3}.k^5} &&\leqslant \frac{f_S(100.(k+1))}{\frac{40}{3}.k^5}, \\
\lim_{k\to\infty} \frac{f_S(100.k)}{\frac{40}{3}.k^5} &\leqslant \lim_{k\to\infty} \frac{f_S(100.k+i)}{\frac{40}{3}.k^5} &&\leqslant \lim_{k\to\infty} \frac{f_S(100.(k+1))}{\frac{40}{3}.k^5}, \\
\lim_{k\to\infty} \frac{f_S(100.k)}{\frac{40}{3}.k^5} &\leqslant \lim_{k\to\infty} \frac{f_S(100.k+i)}{\frac{40}{3}.k^5} &&\leqslant \lim_{k\to\infty} \frac{f_S(100.(k+1))}{\frac{40}{3}.(k+1)^5}, \\
1 &\leqslant \lim_{k\to\infty} \frac{f_S(100.k+i)}{\frac{40}{3}.k^5} &&\leqslant 1
\end{aligned}$$

$\Longrightarrow \lim_{k\to\infty} \frac{f_S(100.k+i)}{\frac{40}{3}.k^5} = 1$. So $f_S(k)$ behaves for the American denomination set at infinity as $\frac{40.k^5}{3.100^5}$. □

The only amazing numbers for the American denomination set are 1 and 50. This is now very easy. We know $f_S(k)$ is a monotone rising function, and we know from (10) that $f_S(100.k) = 1 + \frac{127}{6}.k + \frac{161}{2}.k^2 + 112.k^3 + 65.k^4 + \frac{40}{3}.k^5$ and therefore $f_S(100) = 293$. Note that for higher k, $f_S(k)$ rises even faster, so all amazing numbers need to be below 100. This uses the algorithm mentioned earlier, a piece of cake to evaluate.

For any given number, you can find a set of coin denominations that makes it amazing. For this question we need to find for every k a set S for which $f_S(k) = k$. You can see that this equation is a violation against units. $f_S(k)$ is the number of possibilities, or the number of lists, while k has the unit "cents." This means the algorithm to construct a set S for every k needs to violate against these units too. This algorithm works only for $k \geqslant 4$. When $k = 1$, $k = 2$ or $k = 3$, $S = \{1, 2, 3\}$.

Data: value k
Result: S
begin
  $S \leftarrow \{1, 2\}$
  while $f_S(k) \neq k$ **do**
    add $f_S(k)$ to S
end

An algorithm to find, given an integer k, a set for which $f_S(k) = k$. Call a_i the ith number that is added to S in the while-loop. Call $S_i = \{1, 2\} \cup \bigcup_{j=1}^{i} \{a_j\}$. Now, let's look at the sequence of a_i.

(i) From (9) we know that $\frac{k}{2} < a_1 = \left\lfloor \frac{k}{2} \right\rfloor + 1 < k$.

(ii) We know that $f_S(k+S_d) = f_{S\setminus\{S_d\}}(k+S_d) + f_S(k) \Longrightarrow f_{S_i}(k) = f_{S_{i-1}}(k) + f_{S_i}(k-a_i) \Longrightarrow a_{i+1} \geqslant a_i$. So we know because all $a_i > \frac{k}{2}$ that $f_{S_i}(k-a_i) = f_{\{1,2\}}(k-a_i)$.

All in all we get $a_{i+1} = a_i + f_{\{1,2\}}(k-a_i)$. Now we separate in 2 cases:

(i) $k - a_i > 2$,

$$\begin{aligned} a_{i+1} = & \quad a_i + \left\lfloor \frac{k-a_i}{2} \right\rfloor + 1 \\ < & \quad a_i + \frac{k}{2} - \frac{a_i}{2} + 1 \\ < & \quad k. \end{aligned}$$

a_i is therefore a strict rising sequence and, if a_i is never $0 < k - a_i \leqslant 2$, it has an upper bound k which is never reached. This is impossible, since all a_i are integers. So there is at least one i for which $0 < k - a_i \leqslant 2$.

(ii) $0 < k - a_i \leqslant 2$,

$$\begin{aligned} a_{i+1} = & \quad a_i + \left\lfloor \frac{k-a_i}{2} \right\rfloor + 1 \\ = & \quad a_i + k - a_i \\ = & \quad k. \end{aligned}$$

Because we know there is at least one i for which $0 < k - a_i \leqslant 2$ and we know that in that case $f_{S_{i+1}}(k) = a_{i+1} = k$, we know the algorithm will end with a correct set S_{i+1}. □

There is no upper bound on the number of amazing numbers a denomination set can have. This can be formulated more exactly.

THEOREM **4** (Consequent amazing numbers)**.** *For any number $z \geqslant 3$, we can construct at least one set with z consequent amazing numbers, namely $S = \{1, z, 2z\} \bigcup_{i=1}^{z} \{i + 4z^2 - 6z + 2\}$. The amazing numbers are 1 and $4z^2 - 5z, 4z^2 - 5z + 1, \ldots, 4z^2 - 4z - 1$.*

Proof. First of all, when we apply Theorem 3 we see that

$$f_S(4z^2 - 5z + j) = f_{S\setminus\{4z^2-6z+2+i\}}(4z^2 - 5z + j) + f_S(z + j - 2 - i), (j = 0 \ldots z - 1).$$

Second, $z + j - 2 - i < 4z^2 - 5z + j$ when $z \geqslant 3$. So as we follow the same line of thought as we did in the previous proof, we can see that

$$f_S(z + j - 2 - i) = f_{\{1,z,2z\}}(z + j - 2 - i).$$

So we get

$$f_S(4z^2 - 5z + j) = f_{S\setminus\{4z^2-6z+3\}}(4z^2 - 5z + j) + f_{\{1,z,2z\}}(z + j - 2 - i).$$

We can apply this for all $4z^2 - 6z + 2 + i$ until we get the equation

$$f_S(4z^2 - 5z + j) = f_{\{1,z,2z\}}(4z^2 - 5z + j) + \sum_{i=1}^{z} f_{\{1,z,2z\}}(z + j - 2 - i).$$

In (8), we already have shown $f_{\{1,z,2z\}}(4z^2 - 5z + j) = 4z^2 - 6z + 2$. We also know from (8) that

$$f_{\{1,z,2z\}}(z + j - 2 - i) = \begin{cases} 0 & \text{if } i > z + j - 2, \\ 1 & \text{if } z + j - 2 \geqslant i > j - 2, \\ 2 & \text{if } j - 2 \geqslant i. \end{cases}$$

Now it is easy to see that when $j \in \{0, 1, 2\}$, $\sum_{i=1}^{z} f_{\{1,z,2z\}}(z+j-2-i) = z+j-2$ and when $j \in \{3, 4, \ldots, z-1\}$, $\sum_{i=1}^{z} f_{\{1,z,2z\}}(z + j - 2 - i) = z + j - 2$.
When we add it all together,

$$\begin{aligned} f_S(4z^2 - 5z + j) &= f_{\{1,z,2z\}}(4z^2 - 5z + j) + \sum_{i=1}^{z} f_{\{1,z,2z\}}(z + j - 2 - i), \\ f_S(4z^2 - 5z + j) &= z + j - 2 + 4z^2 - 6z + 2, \\ f_S(4z^2 - 5z + j) &= 4z^2 - 5z + j. \end{aligned}$$

□

This might seem quite abstract, so I've added 2 graphs (see Figures 1 and 2) showing what this set actually does. It is tilting the small flat areas we've seen in (8) at the right spot to create consequent amazing numbers. These graphs are made for $z = 10$ in Maple™.

Editorial note. A more common approach to partition problems involves generating functions. To illustrate the method, let us show that 1 and 50 are the only amazing numbers for the denomination set $\{1, 5, 10, 25, 50\}$ (and thus for $\{1, 5, 10, 25, 50, 100\}$). To start, it is convenient to restrict the monetary values to multiples of 5 and to think of 5 pennies as another kind of nickel. In this way, the problem of finding the number of ways c_n to make change for the amount $5n$ is the number of solutions in nonnegative integers N, N', D, Q, H of the equation

$$n = N + N' + 2D + 5Q + 10H.$$

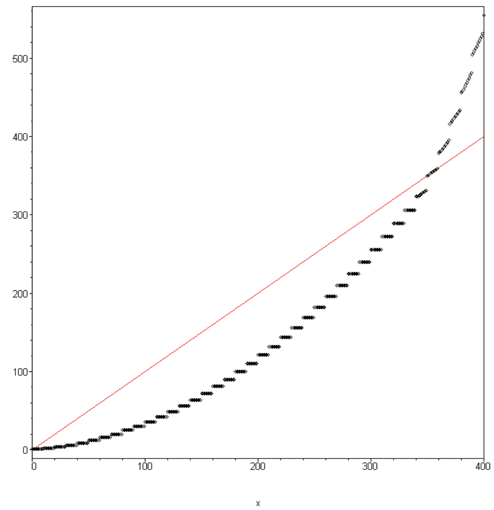

Figure 1: $f_S(k)$

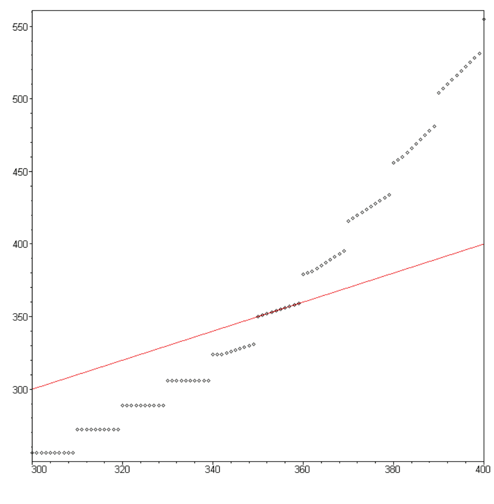

Figure 2: $f_S(k)$

The desired generating function is

$$G(x) = \sum_{n=0}^{\infty} c_n \, x^n = \frac{1}{(1-x)^2(1-x^2)(1-x^5)(1-x^{10})}.$$

Use of partial fractions (easily implemented by any CAS) gives the following result, which is both asymptotic (successive terms are of smaller order of magnitude) and exact (including all five terms leaves no error).

Proposition 1. *The number of ways to give change for* $5n$ *cents using the denomination* *set* $\{N = 5, N' = 5, D = 10, Q = 25, H = 50\}$ *is*

$$c_n = \frac{n^4}{2400} + \frac{19n^3}{1200} + \frac{119n^2}{600} + a_r n + b_r, \quad n \equiv r \pmod{10},$$

where the constants a_r *and* b_r *are given in the following table.*

r	a_r	b_r
0	11/12	1
1	217/240	141/160
2	281/300	6/5
3	217/240	833/800
4	269/300	28/25
5	217/240	41/32
6	11/12	7/5
7	1109/1200	221/160
8	11/12	29/25
9	1061/1200	561/800

More generally, the number of ways of giving change for $m = 5n + k$ *cents, where* $k \in \{0, 1, 2, 3, 4\}$ *is* c_n *as given above.*

In view of this result, a lower bound for the number of ways to give change for m cents is $W(m) = L((m-4)/5)$, where

$$L(x) = \frac{x^4}{2400} + \frac{19x^3}{1200} + \frac{119x^2}{600} + \frac{269x}{300}.$$

(The smallest among the values $a_0, a_1, \ldots, a_9$ is $a_4 = 269/300$ and the values $b_0, b_1, \ldots, b_9$ are all positive.) Because L is convex on $(0, \infty)$, it suffices to check that $W(58) > 58$ to rest assured that any amazing numbers are between 1 and 57. Then use of the exact formula shows that 1 and 50 are the only ones.