A Conjecture on Fields of Extremals with Slopes Diverging

Problem 02-006, *by* L. Bayón, J. Grau, M. Ruiz, and P. Suárez (University of Oviedo, Spain).

In the study of a variety of problems of hydrothermal optimization [1] in which the hydroplants have a pumping capacity, functionals of the type $F(z) = \int_0^T L(t, z(t), z'(t))dt$ appear, and there arises the need to know the boundary conditions for which extremals exist. If the extremals f_λ that satisfy $f_\lambda(0) = a$ and $f'_\lambda(0) = \lambda$ constitute a central field, the possible boundary conditions for the end point are the values of the interval: $(\lim_{\lambda\to-\infty} f_\lambda(T), \lim_{\lambda\to+\infty} f_\lambda(T))$. Hence,

$$\lim_{\lambda\to\pm\infty} f_\lambda(T) = \pm\infty \Longrightarrow \forall\beta \in \mathbb{R},\ \exists\lambda \text{ such that } f_\lambda(T) = \beta.$$

If apart from assuming the monotony of $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ we assume that

$$|L_z(t,z,z')| \le \alpha; \quad z' \le L_{z'}(t,z,z') \le Mz' \ \ \forall z' > 0; \quad z' \ge L_{z'}(t,z,z') \ge mz' \ \ \forall z' < 0,$$

then the well-known Du Bois-Reymond equation [2], satisfied by the extremals, enables us to assure that for each $t \in [0,T]$, $\lim_{\lambda\to\pm\infty} f'_\lambda(t) = \pm\infty$. However, this does not guarantee what may seem very intuitive and which we conjecture is true: $\lim_{\lambda\to\pm\infty} f_\lambda(T) = \pm\infty$.

Conjecture. *Let f_λ be the extremal of the functional $F(z)$ that satisfies the conditions $f_\lambda(0) = a$ and $f'_\lambda(0) = \lambda$. If*

(i) $\lambda_1 < \lambda_2 \Longrightarrow f_{\lambda_1}(t) < f_{\lambda_2}(t) \ \ \forall\, t \in (0,T]$, *and*

(ii) $\lim_{\lambda\to\pm\infty} f'_\lambda(t) = \pm\infty \ \ \forall\, t \in [0,T]$,

then $\lim_{\lambda\to\pm\infty} f_\lambda(T) = \pm\infty$.

Prove or disprove this conjecture. Note that if the collection $\{f'_\lambda\}_{\lambda\in\mathbb{R}}$ is monotone, then the conjecture is true simply by applying the theorem of monotone convergence. If $\{f'_\lambda\}_{\lambda\in\mathbb{R}}$ is not monotone, it appears difficult to prove the conjecture or find a counterexample.

REFERENCES

[1] L. Bayón and P. M. Suárez, *Multiple objective optimization of hydro-thermal systems using Ritz's method*, Math. Probl. Eng., 5 (2000), pp. 379–396.

[2] J. L. Troutman, *Variational Calculus with Elementary Convexity*, Springer-Verlag, New York, 1983.

Status. This problem is open.