## Alternative Proofs of the Bilateral Binomial Theorem

In *Problem* 03-001, Martin Erik Horn asks for alternative proofs of a formula that he discovered, known as the bilateral binomial theorem. This formula reads

$$(1+z)^x = z^y \binom{x}{y} {}_1H_1[(y-x);(y+1);-z], \qquad x, y \in \mathbb{R},\ z \in \mathbb{C},$$

where

$${}_1H_1(a;b;z) = \cdots + \frac{(b-1)(b-2)}{(a-1)(a-2)} z^{-2} + \frac{b-1}{a-1} z^{-1} + 1 + \frac{a}{b}\, z + \frac{a(a+1)}{b(b+1)} z^2 + \cdots .$$

*Note.* In the problem statement of 03-001, the factor $z^y$ was omitted in the initial statement of the identity (eq. (3)). However, in the proof of the identity given by Borwein and Horn (eqs. (8)-(14)), the identity is correctly stated.

*Solution by* G. C. Greubel[1] (Newport News, VA). Consider the bilateral series

$$(1) \qquad {}_1H_1(a;b;x) = \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} x^n.$$

The goal is to find a result for this series in a form similar to that of the standard binomial formula. Equation (1) can be reassembled by the use of the relations

$$(2) \qquad \sum_{n=-\infty}^{-1} f(n) = \sum_{n=0}^{\infty} f(-n-1),$$

$$(3) \qquad (a)_{-n} = \frac{(-1)^n}{(1-a)_n},$$

$$(4) \qquad (a)_{-n-1} = \frac{(-1)^n}{(1-a)(2-a)_n}.$$

This yields

$$(5) \qquad {}_1H_1(a;b;x) = {}_2F_1(a,1;b;x) + \frac{(1-b)}{(1-a)x} {}_2F_1\left(2-b, 1; 2-a; \frac{1}{x}\right).$$

The hypergeometric series integral representation is given by

$$(6) \qquad {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a} du.$$

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With the use of this integral we then have

$$\frac{1}{b-1}{}_1H_1(a;b;x) = \int_0^1 (1-u)^{b-2}(1-xu)^{-a}du$$

$$-\frac{1}{x}\int_0^1 (1-u)^{-a}\left(1-\frac{u}{x}\right)^{b-2}du \tag{7}$$

$$= I_1 - \frac{1}{x}I_2. \tag{8}$$

The evaluation of these integrals will follow. In $I_1$ let $u = \frac{t}{\sqrt{x}}$ to obtain

$$I_1 = \frac{1}{\sqrt{x}}\int_0^{\sqrt{x}} (1-\sqrt{x}t)^{-a}\left(1-\frac{t}{\sqrt{x}}\right)^{b-2} dt. \tag{9}$$

In the second integral let $u = \sqrt{x}t$ to obtain

$$I_2 = \sqrt{x}\int_0^{1/\sqrt{x}} (1-\sqrt{x}t)^{-a}\left(1-\frac{t}{\sqrt{x}}\right)^{b-2} dt. \tag{10}$$

When use is made of equations (8), (9), and (10) we then have

$$\frac{\sqrt{x}}{b-1}{}_1H_1(a;b;x) = \int_0^{\sqrt{x}} (1-\sqrt{x}t)^{-a}\left(1-\frac{t}{\sqrt{x}}\right)^{b-2} dt$$

$$-\int_0^{1/\sqrt{x}} (1-\sqrt{x}t)^{-a}\left(1-\frac{t}{\sqrt{x}}\right)^{b-2} dt \tag{11}$$

$$= \int_{1/\sqrt{x}}^{\sqrt{x}} (1-\sqrt{x}t)^{-a}\left(1-\frac{t}{\sqrt{x}}\right)^{b-2} dt. \tag{12}$$

In order to evaluate this integral make the transformation $t = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)u + \frac{1}{\sqrt{x}}$ to obtain

$$\frac{1}{b-1}{}_1H_1(a;b;x) = (1-x)^{-a}\left(1-\frac{1}{x}\right)^{b-1}\int_0^1 u^{-a}(1-u)^{b-2}du$$

$$= (1-x)^{-a}\left(1-\frac{1}{x}\right)^{b-1} B(1-a, b-1), \tag{13}$$

or

$${}_1H_1(a;b;x) = \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}\frac{(1-x)^{b-a-1}}{(-x)^{b-1}}. \tag{14}$$

Whence we have the relation sought in terms of a binomial-like formula.

The problem asks us to show a particular evaluation of equation (14). Let $a = y - x$, $b = y + 1$, and $x = -z$ for which equation (14) becomes

$$ {}_1H_1(y-x; y+1; -z) = \frac{\Gamma(x-y+1)\Gamma(y+1)}{\Gamma(x+1)} \frac{(1+z)^x}{(z)^y}, \tag{15} $$

or

$$ \binom{x}{y} {}_1H_1(y-x; y+1; -z) = \frac{(1+z)^x}{z^y}. \tag{16} $$

This is the desired result. When $z = 1$ this reduces to

$$ \binom{x}{y} {}_1H_1(y-x; y+1; -1) = 2^x, \tag{17} $$

which is the example given in equation (2) of the problem proposed.

It is noteworthy to use the derived formulas to show more particular values. Some of these values are given by

$$ \binom{x}{y} {}_1H_1(y-x; y+1; 1) = 0, \tag{18} $$

$$ \binom{x}{y} {}_1H_1\left(y-x; y+1; \frac{1}{2}\right) = (-1)^y 2^{y-x}, \tag{19} $$

$$ \binom{x}{x+y} {}_1H_1\left(y; x+y+1; -\frac{1}{2}\right) = 2^y 3^x, \tag{20} $$

$$ {}_1H_1(-x; 1; -z) = (1+z)^x, \tag{21} $$

$$ \binom{x}{y} {}_1H_1(y-x; y+1; 2) = \frac{(-1)^{x-y}}{2^y}, \tag{22} $$

$$ \binom{2x}{2y} {}_1H_1(2y-2x; 2y+1; 2) = \frac{1}{4^y}, \tag{23} $$

$$ \binom{2x}{2} {}_1H_1(2-2x; 3; 2) = \frac{1}{4}, \tag{24} $$

$$ {}_1H_1(-2x; 1; 2) = 1, \tag{25} $$

$$ {}_1H_1(0; x+1; -z) = {}_1H_1\left(-x; 1; \frac{-1}{z}\right) = (1+z)^x. \tag{26} $$

The bilateral binomial series can be used to obtain particular values when summed. One such example is given here. By starting with equation (16) let $y \to 2y$ and $x \to y$ to obtain the form

$$\binom{y}{2y} {}_1H_1(y; 2y+1; -z) = \left(\frac{2}{z^2}\right)^y \left(\frac{1+z}{2}\right)^y. \tag{27}$$

By adding this equation to one of the same with $z$ replaced by $-z$ we have

$$\sum_{r=0}^{1} \binom{y}{2y} {}_1H_1(y; 2y+1; (-1)^r z) = \left(\frac{2}{z^2}\right)^y \left[\left(\frac{1+z}{2}\right)^y + \left(\frac{1-z}{2}\right)^y\right]. \tag{28}$$

From this form of the right-hand side it is evident that when $z$ is assigned the value $\sqrt{5}$ then the resulting form becomes

$$\sum_{r=0}^{1} \binom{y}{2y} {}_1H_1(y; 2y+1; (-1)^r \sqrt{5}) = \left(\frac{2}{5}\right)^y L_y, \tag{29}$$

where $L_m$ is the $m^{\text{th}}$ Lucas number. A similar relation can be derived for the Fibonacci numbers, namely,

$$\binom{y}{2y+1} \sum_{n=0}^{1} {}_1H_1(y+1; 2y+2; (-1)^n \sqrt{5}) = \left(\frac{2}{5}\right)^y F_y. \tag{30}$$

Infinite sums may also be derived. An example will be shown here. Again start with equation (16); let $y \to \alpha n$ to obtain

$$\binom{x}{\alpha n} {}_1H_1(\alpha n - x; \alpha n + 1; -z) = \frac{(1+z)^x}{z^{\alpha n}}. \tag{31}$$

Now summing over $n$ from zero to infinity provides the result

$$\sum_{n=0}^{\infty} \binom{x}{\alpha n} {}_1H_1(\alpha n - x; \alpha n + 1; -z) = \frac{z^\alpha (1+z)^x}{z^\alpha - 1}. \tag{32}$$

Some other examples are given by

$$\sum_{n=0}^{\infty} \binom{n}{y} {}_1H_1(y - n; y + 1; z) = \frac{(-1)^y}{z^{y+1}}, \tag{33}$$

$$\sum_{n=0}^{\infty} \binom{2x}{2n} {}_1H_1(2n - 2x; 2n + 1; 2) = \frac{4}{3}, \tag{34}$$

$$\sum_{n=0}^{\infty} \binom{x}{2n} {}_1H_1(2n - x; 2n + 1; \sqrt{5}) = 5 \cdot 2^{x-2} \beta^x, \tag{35}$$

where $\beta = (1-\sqrt{5})/2$.

*Editorial note.* Interest abides in the history of the bilateral binomial theorem. In a recent manuscript [2], Berndt and Chu write that it would seem that Ramanujan had discovered the identity, "but we are unaware of any mention of it by Ramanujan in his papers or notebooks." They then go on to derive Horn's theorem by starting with a theorem of Dougall [1, p. 110],

$$ {}_2H_2(a,b;c,d;1) = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, $$

replacing $b$ by $dz$ and then letting $d \to \infty$.

## REFERENCES


[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] B. C. Berndt and W. Chu, *Two entries on well-poised bilateral series in Ramanujan's lost notebook*, Available at
`http://www.math.uiuc.edu/~berndt/articles/poised.pdf`.