Dictionaries Are Like Timepieces

Problem 03-002, *by* JONATHAN BORWEIN (Simon Fraser University, Burnaby, BC, Canada).

Samuel Johnson observed that dictionaries were like clocks; the best would not run true and the worst were better than none. The same is true of tables and databases. In [1] Michael Berry reveals that on a desert island he "would give up Shakespeare in favor of Prudnikov, Brychkov and Marichev." That excellent compendium [3] contains (entry 9, page 750)

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{k^2\left(k^2-kl+l^2\right)}=\frac{\pi^{\propto}\sqrt{3}}{30}, \tag{1}$$

where the "$\propto$" is probably "4."

Integer relation methods [2] suggest that no reasonable value of $\propto$ works. What is intended in formula (1)?

Note that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2}{n^2\left(n^2-mn+m^2\right)}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2}{n^2\left(n^2+mn+m^2\right)}\\ =\sum_{\substack{m,n\in\mathbb{Z}\\ mn\neq 0}}\frac{1}{n^2\left(n^2+mn+m^2\right)}=6\,\zeta(4),\end{aligned}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}\frac{1}{nm\left(n^2+mn+n^2\right)}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^2\left(n^2+mn+m^2\right)}=\frac{13}{12}\,\zeta(4),$$

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^2\left(n^2+mn+m^2\right)}&=\frac{2}{\sqrt{3}}\,\mathrm{Im}\sum_{n=1}^{\infty}\frac{\Psi\left(1+n\frac{-1+i\sqrt{3}}{2}\right)}{n^3}\\ &=\frac{2}{\sqrt{3}}\,\mathrm{Im}\int_0^{\infty}\frac{\mathrm{Li}_3\left(e^{\left(\frac{-1+i\sqrt{3}}{2}\right)t}\right)}{e^t-1}\,dt\\ &=1.004457198201574027554140 25\ldots.\end{aligned}$$

Here Ψ and Li_3 are the *digamma* and *trilogarithm*, respectively [2].

REFERENCES

[1] M. BERRY, *Why are special functions so special?*, Phys. Today, 54 (2003), pp. 11–12; also available online from http://www.physicstoday.org/pt/vol-54/iss-4/p11.html.

[2] J.M. Borwein and D.H. Bailey, *Mathematics by Experiment*, A. K. Peters Ltd., Wellesley, MA, 2003.

[3] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series*, Volumes 1 and 2, Gordon and Breach, New York, 1986.

Status. This problem appears in [2]. It is unsolved, though the listed identities are known. Indeed one may ask, Is it the left side or the right side of (1) that is in need of corrections?