*Solution of Problem* 03-004 *by* Vinicius-Nicolae-Petre Anghel (AECL Chalk River Laboratory, Chalk River, Ontario, Canada).

As a start, we evaluate the finite product

$$\prod_{n=1}^{N} \frac{y^{2^{-n}}+1}{z^{2^{-n}}+1}.$$

First

$$\left(y^{2^{-N}}-1\right) \prod_{n=1}^{N} \left(y^{2^{-n}}+1\right) = y-1.$$

The proof consists in applying $x^2-1=(x-1)(x+1)$ repeatedly to get

$$\begin{aligned} y^{2^{-N+1}}-1 &= \left(y^{2^{-N}}-1\right)\left(y^{2^{-N}}+1\right), \\ y^{2^{-N+2}}-1 &= \left(y^{2^{-N+1}}-1\right)\left(y^{2^{-N+1}}+1\right), \\ \vdots \quad & \qquad \vdots \qquad\qquad \vdots \\ y-1 &= \left(y^{2^{-1}}-1\right)\left(y^{2^{-1}}+1\right). \end{aligned}$$

Putting the results for the numerator and denominator products together, we obtain

$$\prod_{n=1}^{N} \frac{y^{2^{-n}}+1}{z^{2^{-n}}+1} = \frac{y-1}{z-1}\,\frac{z^{2^{-N}}-1}{y^{2^{-N}}-1}.$$

All that remains is to take the limit $N \to \infty$:

$$\prod_{n=1}^{\infty} \frac{y^{2^{-n}}+1}{z^{2^{-n}}+1} = \frac{y-1}{z-1} \lim_{N\to\infty} \frac{z^{2^{-N}}-1}{y^{2^{-N}}-1} = \frac{y-1}{z-1}\, \frac{\frac{d}{d\alpha}\left(z^{\alpha}-1\right)\Big|_{\alpha=0}}{\frac{d}{d\alpha}\left(y^{\alpha}-1\right)\Big|_{\alpha=0}}.$$

Therefore

$$\prod_{n=1}^{\infty} \frac{y^{2^{-n}}+1}{z^{2^{-n}}+1} = \frac{y-1}{z-1}\,\frac{\ln z}{\ln y},$$

and for $y=e$ we have

$$\prod_{n=1}^{\infty} \frac{e^{2^{-n}}+1}{z^{2^{-n}}+1} = \frac{e-1}{z-1} \ln z.$$

Along the same lines one can prove

$$\prod_{n=1}^{\infty} \frac{y^{2\cdot 3^{-n}}+y^{3^{-n}}+1}{z^{2\cdot 3^{-n}}+z^{3^{-n}}+1} = \frac{y-1}{z-1}\,\frac{\ln z}{\ln y},$$

or, more generally,

$$\prod_{n=1}^{\infty} \frac{y^{(k-1)\cdot k^{-n}} + y^{(k-2)\cdot k^{-n}} + \cdots + y^{k^{-n}} + 1}{z^{(k-1)\cdot k^{-n}} + z^{(k-2)\cdot k^{-n}} + \cdots + z^{k^{-n}} + 1} = \frac{y-1}{z-1} \frac{\ln z}{\ln y}.$$

The proof is entirely similar, using

$$y^k - 1 = (y-1)\left(y^{k-1} + y^{k-2} + y^{k-3} + \cdots + y + 1\right)$$

instead of

$$y^2 - 1 = (y-1)(y+1).$$

Another result that can be proved in the same manner is

$$\prod_{n=1}^{\infty} \left(y^{(k-1)\cdot k^{-n}} - y^{(k-2)\cdot k^{-n}} + \cdots - y^{k^{-n}} + 1\right) = \frac{1}{2}(y+1)$$

for $k$ odd. Here one uses

$$y^k + 1 = (y+1)\left(y^{k-1} - y^{k-2} + y^{k-3} + \cdots - y + 1\right).$$

Yet another limit proved similarly is

$$\prod_{n=1}^{\infty} \left(y^{2^{-n+1}} - y^{2^{-n}} + 1\right) = y^2 + y + 1.$$

The telescoping factor by which to multiply the finite product is $(y^{2^{-N+1}} + y^{2^{-N}} + 1)$, and the identity to use is

$$y^4 + y^2 + 1 = \left(y^2 + y + 1\right)\left(y^2 - y + 1\right).$$