A Theorem on Entire Functions of Finite Order with Positive Taylor Coefficients

Solution of Problem 04-004 *by the proposer.*

We shall prove that for $f \in A_\rho$, where $0 \le \rho < \infty$,

$$\lim_{r\to\infty} T(r) = 1,$$

independently of the order ρ. Hence $c_\rho^- = c_\rho^+ = 1$. The proof is in two parts.

1. $\liminf_{r\to\infty} T(r) \ge 1$.

Proof. Suppose that f is a transcendental entire function with positive Taylor coefficients and of finite or infinite order. Thus $f(z) := \sum a_n z^n$, where $a_n > 0$ for all n. Define

$$f_0(z) := f(z); \qquad f_1(z) := zf'(z) = \sum n a_n z^n; \qquad f_2(z) := zf_1'(z) = \sum n^2 a_n z^n.$$

Then

$$T(r) = \frac{f_0(z) f_2(z)}{(f_1(z))^2} - \frac{f_0(z)}{f_1(z)}, \tag{1}$$

and, by Cauchy's inequality,

$$\frac{f_0(r) f_2(r)}{(f_1(r))^2} > 1, \qquad r > 0.$$

Also, since

$$r\left(\frac{f_0(r)}{f_1(r)}\right)' = 1 - \frac{f_0(r) f_2(r)}{(f_1(r))^2} < 0, \qquad r > 0,$$

we conclude that f_0/f_1 is monotone decreasing and approaches the limit zero. Otherwise, there exists $a > 0$ such that $f_0(r)/f_1(r) > a$ for all $r > 0$. Hence $f'(r)/f(r) < \dfrac{1}{ar}$, and integration yields $f(r) = O(r^{1/a})$. In this case f is a polynomial, not a transcendental function. Hence $\liminf_{r\to\infty} T(r) \ge 1$. □

2. $\limsup_{r\to\infty} T(r) \le 1$.

Proof. We shall use Karamata's concept of *regularly varying* functions.

Definition 1. *A positive measurable function k belongs to the class K_ρ of regularly varying functions with index $\rho \in \mathbb{R}$ if the relation*

$$\lim_{x\to\infty} \frac{k(\lambda x)}{k(x)} = \lambda^\rho \tag{0.1}$$

holds for each $\lambda > 0$.

Remark 1. *Moreover, if $k \in K_\rho$ then* (0.1) *holds uniformly on each compact λ-set in* $(0, \infty)$ [1, p. 6].

The theory of regular variation is well developed and has many applications in analysis, number theory, probability, etc. For general theory and applications, see [1] and [2]. Setting

$$t_0(r) := f_1(r)/f_0(r); \qquad t_1(r) = f_2(r)/f_1(r),$$

(1) gives

$$T(r) = \frac{t_1(r)}{t_0(r)} - \frac{1}{t_0(r)}.$$

As we have already proved, $t_0(r)$ and $t_1(r)$ are monotone increasing to infinity; hence the second term in (2) does not affect further estimations of $T(r)$.

The proof of our second assertion depends entirely on the following two results.

Lemma 1. *Let g be a positive function with the property*

$$\limsup_{r\to\infty} \frac{\log g(x)}{\log x} = \rho, \qquad 0 \le \rho < \infty.$$

Then there exists a regularly varying function $k \in K_\rho$ such that $g(x) \le k(x)$ and

$$\limsup_{x\to\infty} \frac{k(x)}{g(x)} = 1.$$

This is a known theorem on approximation by a regularly varying function [1, p. 81].

Lemma 2. *If $f \in A_\rho$, then*

$$\limsup_{r\to\infty} \frac{\log t_0(r)}{\log r} = \rho.$$

See [3, section IV, chapter 1, problem 53].

Combining the two lemmas we find that for $f \in A_\rho$ there exists a regularly varying function $k \in K_\rho$ such that

$$t_0(r) \le k(r); \qquad \limsup_{r\to\infty} \frac{k(r)}{t_0(r)} = 1. \tag{2}$$

Hence for arbitrary $C > 1$ we find

$$\frac{k(r)}{C} < t_0(r) \le k(r), \qquad r > r_0(C), \tag{3}$$

and, consequently, for each $\lambda > 1$,

$$\frac{k(\lambda r)}{C} < t_0(\lambda r) \le k(\lambda r), \qquad r > r_0(C). \tag{4}$$

From (3) and (4) it follows that

$$\frac{t_0(\lambda r)}{t_0(r)} < C\,\frac{k(\lambda r)}{k(r)}, \qquad r > r_0(C). \tag{5}$$

Hence by Definition 1 and Remark 1 we find that

$$\frac{t_0(\lambda r)}{t_0(r)} < C^2\lambda^\rho, \qquad r > r_1(C), \tag{6}$$

holds for each $\lambda > 1$.

Another assertion is of importance.

Lemma **3.** *For each $\lambda > 1$ and $r > 0$, we have*

$$t_i(r)\log\lambda < \log\frac{f_i(\lambda r)}{f_i(r)} < t_i(\lambda r)\log\lambda, \qquad i = 0, 1.$$

Proof. Indeed, since $t_i(r)$ is monotone increasing for $r > 0$, we get

$$t_i(r)\log\lambda = t_i(r)\int_r^{\lambda r}\frac{du}{u} < \int_r^{\lambda r}\frac{t_i(u)\,du}{u} = \log\frac{f_i(\lambda r)}{f_i(r)} < t_i(\lambda r)\log\lambda.$$

□

Since $t_0 = f_1/f_0$, combining Lemma 3 with the inequality in (6), it follows that

$$\begin{aligned} t_1(r)\log\lambda &< \log\frac{f_1(\lambda r)}{f_1(r)} < \log(C^2\lambda^\rho) + \log\frac{f_0(\lambda r)}{f_0(r)} \\ &< \log(C^2\lambda^\rho) + t_0(\lambda r)\log\lambda \\ &< \log(C^2\lambda^\rho) + (C^2\lambda^\rho\log\lambda)t_0(r), \qquad r > r_1(C). \end{aligned} \tag{7}$$

Since $t_0(r) \uparrow \infty$, choosing $\lambda = C$ if $\rho = 0$ and $\lambda = C^{1/\rho}$ for $\rho > 0$ in (7), we get

$$\frac{t_1(r)}{t_0(r)} < o(1) + C^3, \qquad r \to \infty.$$

Because $C > 1$ is arbitrary, by (2) we finally conclude that

$$\limsup_{r \to \infty} T(r) \leq 1,$$

and the proof is complete. □

Remark **2.** *The above theorem is also valid for transcendental entire functions of finite order and with nonnegative Taylor coefficients.*

REFERENCES

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, UK, 1989.

[2] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, New York, 1976.

[3] G. Pólya and G. Szegö, *Problems and Theorems in Analysis, Vol.* II, Springer-Verlag, Berlin, 1976.