Also Eisenstein

In Problem 05-002, Jonathan Borwein[1] (Dalhousie University, Halifax, NS, Canada) asked for a closed form evaluation of

$$\zeta_G(N) = \sum_{z\in\mathbb{Z}[i]}{}' \frac{1}{z^N} = \sum_{m,n}{}' \frac{1}{(m+ni)^N}$$

for positive integer $N > 1$. The primed summation signifies that $(m, n) = (0, 0)$ is excluded, and "closed form" means the product of a simply determined rational number and a power of a special function value.

Solution by the proposer. It is evident by symmetry that $\zeta_G(N) = 0$ in case $N \equiv 1 \pmod 4$ or $N \equiv 2 \pmod 4$. For the latter case, note that $z \to iz$ maps the lattice of Gaussian integers onto itself by a 90° rotation centered at the origin. Thus the value of the sum is preserved; on the other hand, $(iz)^N = -z^N$, so the value changes sign.

To deal with the remaining case $N \equiv 0 \pmod 4$, recall that the sums in question have another name—they are the *Eisenstein series*, which we write as

$$S_k = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+ni)^k}, \qquad k \geq 2.$$

These sums are connected with the *Weierstrass* $\wp$ *function* given by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{(m+ni-z)^2} - \frac{1}{(m+ni)^2} \right\}.$$

The Laurent expansion of the latter is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) S_{2n+2}\, z^{2k} = \frac{1}{z^2} + 3S_4 z^2 + 7S_8 z^6 + 11 S_{12} z^{10} + \cdots.$$

Let

$$g_2 = 60S_4, \qquad g_3 = 140S_6 = 0.$$

In the brief appendix, some standard relations for elliptic functions are used to establish $g_2 = 4L^4$, where L is the *lemniscate constant* given by

$$L = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \sqrt{2}K\left(\tfrac{1}{\sqrt{2}}\right) = \frac{1}{2}B\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = \frac{1}{2\sqrt{2\pi}}\Gamma^2\left(\tfrac{1}{4}\right).$$

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(As is standard, $K(k)$ is the complete elliptic integral of the first kind and $B(x, y)$ is the beta function.) Hence $\zeta_G(4) = S_4 = g_2/60$ is a rational multiple of a special function value.

The $\wp$ function satisfies the differential equation

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3,$$

which upon differentiation yields

$$\wp''(z) = 6\wp^2(z) - \frac{g_2}{2}. \tag{1}$$

To simplify the notation, write the Laurent expansion as

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} c_n\, z^{4n-2}, \qquad c_n = (4n-1)S_{4n},$$

and note that

$$\frac{g_2}{2} = 10c_1.$$

Substitution into (1) yields

$$6 + \sum_{k=1}^{\infty}(4k-2)(4k-3)c_k w^k = 6\left(1 + \sum_{k=1}^{\infty} c_k w^k\right)^2 - 10c_1 w, \qquad w = z^4.$$

Comparing coefficients gives

$$[(4n-2)(4n-3) - 12]c_n = 6\sum_{k=1}^{n-1} c_k c_{n-k}, \qquad c_1 = \frac{g_2}{20} = \frac{L^4}{5},$$

so writing $c_n = p_n L^{4n}$ leads to

$$p_n = \frac{3}{(4n+1)(2n-3)}\sum_{k=1}^{n-1} p_k p_{n-k}, \qquad p_1 = \frac{1}{5}.$$

Thus the sequence (p_n) begins

$$p_1 = \frac{1}{5}, \quad p_2 = \frac{1}{75}, \quad p_3 = \frac{2}{4875}, \quad p_4 = \frac{1}{82875}, \quad p_5 = \frac{2}{6215625},$$
$$p_6 = \frac{2}{242409375}, \quad p_7 = \frac{4}{19527421875}, \quad p_8 = \frac{223}{444815433203125}.$$

Finally,

$$\zeta_G(4n) = S_{4n} = \frac{p_n L^{4n}}{4n-1},$$

so we have written each nonvanishing zeta sum as the product of a simply determined rational number and a power of a special function value as desired. For a related development, see [2, pp. 167–170].

Appendix. The following formulas relating the lemniscate constant to special values of the theta functions go back to Gauss and are quoted in [3, p. 18]:

$$L = \pi\theta_2^2(e^{-\pi}) = \pi\theta_4^2(e^{-\pi}) = \frac{\pi\sqrt{2}}{2}\theta_3^2(e^{-\pi}).$$

Use of the formula for the discriminant $\Delta = g_2^3 - 27g_3^2$ in terms of theta functions [1, p. 179] gives

$$\begin{aligned} g_2^3 = \Delta &= 16\pi^{12}\{\theta_2(q)\theta_3(q)\theta_4(q)\}^8, \qquad q = e^{-\pi} \\ &= 16\pi^{12}(L/\pi)^4(L/\pi)^4(\sqrt{2}L/\pi)^4 \\ &= 2^6 L^{12}, \end{aligned}$$

so $g_2 = 4L^4$.

REFERENCES

[1] J. W. Armitage and W. F. Eberline, *Elliptic Functions*, Cambridge University Press, Cambridge, UK, 2006.

[2] J. M. Borwein, D. Bailey, and R. Girgensohn, *Experimentation in Mathematics*, A K Peters, Ltd., Natick, MA, 2004.

[3] J. Todd, *The lemniscate constants*, Collection of articles honoring Alston S. Householder, Comm. ACM, 18 (1975), pp. 14–19; corrigendum, ibid. 18 (1975), p. 462.