Average Partial Order Width of the Divisor Poset

Problem 05-003, *by* Marko Riedel (EDV, Neue Arbeit gGmbH, Stuttgart, Germany).

The divisors of a positive integer n form a partially ordered set (poset). Let $g(n)$ be the size of the largest antichain (set of pairwise incomparable elements). Compute the asymptotic expansion of $G(n)$, the average order of $g(n)$, i.e.,

$$G(n) = \frac{1}{n} \sum_{k=1}^{n} g(k).$$

The size of the largest antichain is also called the partial order width. In the case of the divisor poset, the partial order width is the width of the Hasse diagram of the poset. The Hasse diagram of the divisor poset of $n = 24$ is shown in Figure 1. The partial order width of this poset is 2, i.e., $g(24) = 2$.

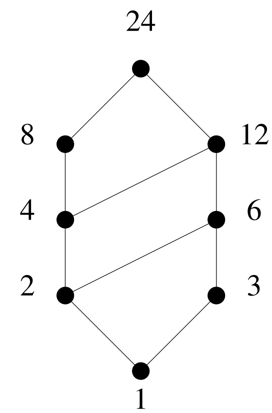

Figure 1. *Hasse diagram of the divisor poset of* $n = 24$.

The divisor poset D_n of n is a product of chains, one for each prime divisor of n. Suppose the prime factorization of n is

$$n = \prod_{j=1}^{m} p_j^{v_j}.$$

Then we have

$$D_n \cong \prod_{j=1}^{m} [v_j], \qquad \text{where} \qquad [v] = \{0, 1, \dots, v\}.$$

This is an example of a "PECK" poset which is "Sperner," so its largest antichain is its largest rank level. The rank generating function for the above factorization is

$$\prod_{j=1}^{m} \left(1 + x + x^2 + \cdots + x^{v_j}\right).$$

The middle coefficient of this polynomial (or the two middle coefficients) yields $g(n)$. This is a central binomial coefficient when n is squarefree.

CONJECTURE. *The average order of the partial order width of the divisor poset D_n of n is $\theta(\log n)$.*

Evidence for this conjecture is shown in Figure 2, where we have plotted $G(n)$.

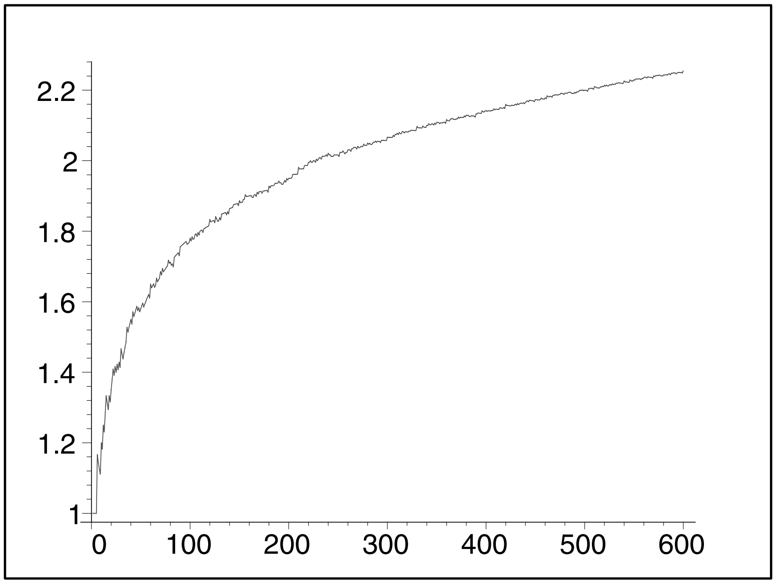

Figure 2. *Average order of the partial order width of the divisor poset up to $n = 600$.*

The following set of MAPLE procedures maybe useful in further investigation of $G(n)$.

```
 rgf := proc(n)
    local f, p, res;
    p := 1;
    for f in op(2, ifactors(n)) do
        p := p * sum(x^j, j=0 .. op(2, f));
    od;
    expand(p);
end proc;

g := proc(n)
    local p;
    p := rgf(n);
    coeff(p, x, iquo(degree(p, x),2));
end proc;
```

```
pts := proc(n)
    local l, k, s;
    l := [[1, 1]]; s:= 1;
    for k from 2 to n do
        s := s + g(k);
        l := [op(l), [k, s/k]];
    od;
    l;
end proc;

plotsetup
    (ps,plotoutput=‘dposetdata.ps‘,
     plotoptions=‘portrait,
     width=4in,height=3in‘);
plot(pts(600));
```

This problem was first posed on `sci.math.research` under the title “Divisors and Antichains.”

Status. This problem is open.